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Chern-Simons Theory and Knot Polynomials

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Abstract: In this review, we discuss one of Edward Witten's Fields Medal winning works – *Quantum Field Theory and the Jones Polynomial.* Mathematical machinery such as holonomy, homotopy, and bundles are first introduced as the necessary background for this review. Next, the essential notions of knot theory are reviewed prior to the discussion of Chern-Simons gauge theory. We then review Witten's groundbreaking ideas on the emergence of knot and link invariants, and three-manifold invariants from Chern-Simons theory. Finally, we discuss several explicit computations of the invariants.

Contents

1 Introduction

Physics and geometry have been the two inseparable fields in our attempt of systematising intuitions we have about the structure of space and time. These two interrelated fields yield a plethora of surprising results over the past decades. In particular, quantum field theory (QFT) and knot theory have been brought together in Edward Witten's 1989 groundbreaking work, which was hugely motivated by the two problems proposed by Michael Atiyah in the 1987 Hermann Weyl Symposium [1]. The first problem was to give a physical interpretation to Donaldson theory. The second problem was to find an intrinsically 3-dimensional definition of the Jones polynomial of knot theory.

Donaldson theory [2] is essential to understanding geometry in four dimensions, which is our physical dimension at least macroscopically. Using the moduli spaces of solutions of the self-dual Yang-Mills equations, the theory addressed the issues related to the natural topological invariants of a 4-manifold, namely the second homology group and its intersection form. Its interpretation in terms of QFT was then shown by Floer [3] and Witten [4].

Unlike the Donaldson theory, where its connection with QFT was not obvious, the knot polynomials have been intimately connected with two dimensional many body physics in a bewildering variety of ways, from solvable lattice models [5], solutions of the Yang-Baxter equation [6] to conformal field theory [7–9]. The challenge on the physical side has been to find the unifying themes for the diversity of the knot and link polynomials. On the mathematical side, the puzzle was that the Jones polynomial and its generalisations, despite being invariants of a 3-dimensional case, have no intrinsically 3-dimensional definitions.

Now, what is a topological invariant? It is simply a term referred by mathematicians to quantity computed from a manifold as a topological space (perhaps with a smooth structure) without a choice of metric and is conserved under continuous deformation (essentially invariant up to diffeomorphism). To physicists, the physical meaning is really *general covariant*. Under the influence of relativity theories, one would naturally describe a generally covariant QFT by introducing a metric (with no *a priori* choice) and integrate all over the metric. This means that the theory depends on the metric, which is a dynamical variable.

In the 1980s the seminal works of Donaldson, Jones, Floer, and Gromov [10] brought about a different perspective in which general covariance can be realised without the metric. It was exactly the theme in Witten's 1989 paper [11] in which a solvable 3-dimensional model where both general covariance and gauge invariance are realised at the full quantum level. He proved that the Chern-Simons (CS) gauge theory in the presence of Wilson loops leads to topological invariant of 3-manifolds that are closely related to Jones polynomial of knots. In particular, he found a way to work out the exact solution of CS theory in terms of a 2-dimensional conformal field theory (the Wess-Zumino-Witten (WZW) model), which is deeply related to the knot and 3-manifold invariants. Since then both CS and knot theories have been intensively studied, making important progress via applications of field theory methods.

In this review, we will outline and (re-)construct Witten's ideas. We first consider the mathematical machinery needed to describe the results in Witten's paper. In the next section, we present the CS theory and its important properties. In the last section we discuss a series of methods on solving CS theory on a 3-manifold and consequently show how the knot and manifold invariants arise naturally from the theory.

2 Mathematical Tools

In this section, we present some mathematical concepts which we will use in our discussion. The topics covered here are holonomy, homotopy, bundles and connections, based on [12–14].

2.1 Holonomy

A loop is a fundamental mathematical object, which can be extended to knots and links in later section.

Definition 1. A **loop** is a closed curve

$$\begin{aligned} C : \mathbf{S}^1 &\to \mathcal{M} \\ t \in [0,1] &\mapsto p = C(t), \quad C(0) = C(1). \end{aligned} \tag{2.1}$$

Consider a set of loops passing through a point p in the manifold $\mathcal{M}$ of dimension n. We parallelly transport a vector $X \in T_p\mathcal{M}$ round a loop C through p and end up with a new vector $X_C \in T_p\mathcal{M}$. This induces a linear transformation M_C, which is called the **holonomy** at p of the connection for the loop $C(t)$, such that

$$\begin{aligned} M_C : T_p\mathcal{M} &\to T_p\mathcal{M} \\ X &\mapsto X_C. \end{aligned} \tag{2.2}$$

In other words, a holonomy is a measure of how much the initial and final values of a vector (or more generally a spinor) differ after parallel transport round a loop. With a coordinate basis $\{e_\mu = \frac{\partial}{\partial x^\mu}\}$, the vector $X = X^\mu e_\mu$ transforms to $X_C = X^\mu (M_C)_\mu{}^\nu e_\nu$. The set of all linear transformations M_C for all curves C through p then forms a group Hol_p, the **holonomy group** at p. Clearly this is a subgroup of $GL(n;\mathbb{R})$.

2.2 Homotopy

Now we present the concept of homotopy of loops (generally it can be applied for paths and any topological spaces). Consider two discs in Fig. 1, one has a hole in it, the other does not. The difference between these two discs are clearly illustrated by the loops. In disk Y, any loops can be continuously shrunk to a point. In contrast, some loops in disk X can be shrunk to a point while others cannot. For instance, loop α in disk X cannot be shrunk to a point due to the existence of a hole in it. The idea of homotopy comes in when we say a loop α is equivalent to β through a continuous deformation.

Definition 2. Let α, $\beta : I = [0,1] \to X \subset \mathcal{M}$ be loops at $x_0 \in X$. A **homotopy** between α and β is a continuous map $H : I \times I \to X$ such that

$$\begin{aligned} H(s,0) = \alpha(s), \quad H(s,1) = \beta(s) \quad &\forall s \in I, \\ H(0,t) = H(1,t) = x_0 \quad &\forall t \in I. \end{aligned} \tag{2.3}$$

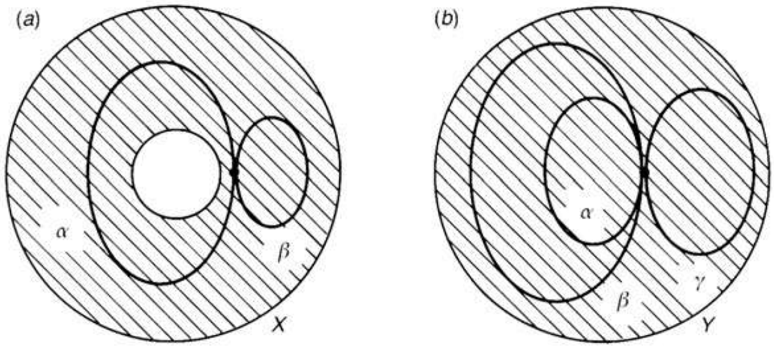


Figure 1: A disc with a hole (a) and without a hole (b). The hole in (a) prevents the loop α from shrinking to a point.

α and β are said to be **homotopic** and we write the homotopic relation as $\alpha \sim \beta$, which is an equivalence relation. The equivalence class of loops is denoted by $[\alpha]$ and is called the **homotopy class** of α. Further, the set of homotopy classes can be endowed with a group structure called the fundamental group.

Homotopy Groups

Definition 3. Let X be a topological space. The set of homotopy classes of loops at $x_0 \in X$, denoted by $\pi_1(X, x_0)$, is called the **first homotopy group** (or more commonly as the **fundamental group**) of X at x_0. The product of homotopy classes $[\alpha]$ and $[\beta]$ is defined by $[\alpha] * [\beta] = [\alpha * \beta]$.

Back to the disk example, notice that there is only one homotopy class associated with Y, whereas each homotopy class in X is characterised by $n \in \mathbb{Z}$, n being the number of times the loop encircles the hole. This means that $n < 0$ if it winds clockwise, $n > 0$ if counterclockwise, and $n = 0$ if the loop does not wind round the hole. This set of integer $\mathbb{Z}$ is an additive group with a geometrical meaning: $n + m$ corresponds to going round the hole first n times and then m times.

We can also generalise homotopy to higher dimensional loops, for instance, spheres $\mathbf{S}^n$ or tori $\mathbf{T}^n$ $(n \geq 2)$. These are known as the n-loops, which we define as follows. Recall that in the fundamental group, the boundary δI of $I = [0, 1]$ is mapped to the base point $x_0 \in X$. We now consider a unit n-cube $I^n = I \times \cdots \times I \, (n \geq 1)$ as $I^n = \{(s_1, \ldots, s_n) | 0 \leq s_i \leq 1, 1 \leq i \leq n\}$ and its boundary $\delta I^n = \{(s_1, \ldots, s_n) \in I^n \mid \text{some } s_i = 0 \text{ or } 1\}$. Then an **$n$-loop** is a continuous map $\lambda : I^n \to X$, where the boundary δI^n is mapped to a point $x_0 \in X$. Naturally, homotopy extends to n-loops.

Definition 4. Let X be a topological space and α, $\beta : I^n \to X$ be n-loops at $x_0 \in X$. The maps α and β are said to be **homotopic**, $\alpha \sim \beta$, if there exists a continuous map $F : I^n \times I \to X$ such that

$$H(s_1, \ldots, s_n, 0) = \alpha(s_1, \ldots, s_n), \quad H(s_1, \ldots, s_n, 1) = \beta(s_1, \ldots, s_n), \\ H(s_1, \ldots, s_n, t) = x_0, \tag{2.4}$$

for $(s_1, \ldots, s_n) \in \delta I^n$, $t \in I$.

Definition 5. Let X be a topological space. The set of homotopy classes of n-loops ($n \geq 1$) at $x_0 \in X$, denoted by $\pi_n(X, x_0)$, is called the nth homotopy group at x_0. $\pi_n(X, x_0)$ is called the **higher homotopy group** if $n \geq 2$.

2.3 Bundles

Many aspects of physics, such as general relativity and gauge theories, can be formulated in the language of bundles. Bundles are also essential to the understanding of topological properties of an underlying manifold. We would like to introduce some notions of the bundle theory here. We assume for simplicity that all the structures that we are going to discuss, such as manifolds and bundles, are *smooth*.

Definition 6. A (differentiable) **bundle** is a triple $(E, \pi, \mathcal{M})$ which consists of the following elements:

1. A differentiable manifold E called the **total space**.
2. A differentiable manifold $\mathcal{M}$ called the **base space**.
3. A surjection $\pi : E \to \mathcal{M}$ called the **projection**.

We often use a shorthand notation $E \xrightarrow{\pi} \mathcal{M}$ or simply E to denote a bundle $(E, \pi, \mathcal{M})$. The definition of a bundle is quite nonrestrictive. For instance, the empty function defines a bundle.

There are often additional structures to the notion of bundle. One of them is called a *fibre* of a bundle.

Definition 7. For each $p \in \mathcal{M}$, the **fibre** of a bundle E over p is the *preimage* of the projection map π, i.e. $F_p = \pi^{-1}(p)$.

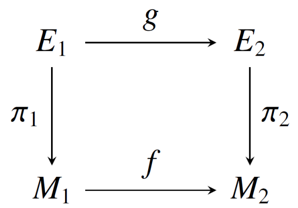
Definition 8. A **section** of the bundle E is a smooth map $s : \mathcal{M} \to E$ such that $\pi \circ s = id_{\mathcal{M}}$.

This terminology comes from the geometric interpretation of the image $s(\mathcal{M}) \subset E$ as a "cross section" of the bundle (total space) E. This notion of section generalises that of functions and fields. It is obvious that $s(p) = s|_p$ is an element of F_p.

Definition 9. The space (set) of all smooth sections of a bundle E over $\mathcal{M}$ is denoted by $\Gamma(\mathcal{M}, E) = \{s \in C^\infty(\mathcal{M}, E) | \pi \circ s = id_{\mathcal{M}}\}$.

For an open set $U \subset \mathcal{M}$, we have a **local section** which is defined only on U ($s : U \to E$) and $\Gamma(U, E)$ denotes the set of local sections on U. Notice that not all fibre bundles admit global sections.

Given two bundles $E_1 \xrightarrow{\pi_1} \mathcal{M}_1$ and $E_2 \xrightarrow{\pi_2} \mathcal{M}_2$ with maps $f : \mathcal{M}_1 \to \mathcal{M}_2$ and $g : E_1 \to E_2$ then the pair (f, g) is said to be a **bundle morphism** if the following diagram commutes, i.e. $\pi_2 \circ g = f \circ \pi_1$.



Two bundles are said to be *isomorphic* if both (f, g) and (f^{-1}, g^{-1}) are bundle morphisms where the inverse of (f, g) exists.

Definition 10. A fibre bundle is a quintuple $(E, \pi, \mathcal{M}, F, G)$ consisting of

1. A total space E, a projection map π, a base space $\mathcal{M}$, together with a **fibre** F, which is diffeomorphic to F_p over $\mathcal{M}$ ($F \cong F_p = \pi^{-1}(p)$, $p \in \mathcal{M}$). The fibre F is also a differentiable manifold and is sometimes called the standard fibre or typical fibre.

2. A Lie group G called the **structure group** that acts on F on the left.

3. An atlas of charts (U_i, φ_i), i.e. a covering of $\mathcal{M}$ by open sets U_i, where i indexes the sets, with maps φ_i called **local trivialisations** of E over $U_i \subset \mathcal{M}$, such that they are diffeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$, where $\varphi_i(p) = (\pi(p), f_i(p))$ for $p \in \pi^{-1}(U_i)$ and $f_i : \pi^{-1}(U_i) \to F$.

4. The **transition functions** which are smooth maps $f_{ij} : U_i \cap U_j \to G$ on $U_i \cap U_j \neq \emptyset$. With a diffeomorphism $f_i : F_p \to F$, the transition functions $f_{ij} \equiv f_i \circ f_j^{-1} : F \to F$ is an element of G.

The transition function f_{ij} is a cocycle which satisfies the conditions: $f_{ii} = 1$ and $f_{ij} = f_{ik} f_{kj}$ for any point in $U_i \cap U_j \cap U_k$. If all the transition functions f_{ij} can be taken to be identity maps, the fibre bundle E is called a **trivial bundle**. In other words, E is trivial if $\mathcal{M}$ is contractible to a point. This result is obtained using the concepts of homotopy maps and pullback bundles as shown in [12]. A trivial bundle is then (diffeomorphic to) a product space, i.e. $E = \mathcal{M} \times F$. A bundle which is not trivial is called *twisted*.

Vector Bundle

Let us choose a field $\mathbb{F}$ = either $\mathbb{R}$ or $\mathbb{C}$ and assume that all vector spaces and linear maps are $\mathbb{F}$-linear. Then we can define a vector bundle as follows.

Definition 11. A **vector bundle** of rank k is a fibre bundle $E = \mathcal{M} \times V$ whose fibre F is a vector space $V = \mathbb{F}^k$ and structure group is $GL(k, \mathbb{F})$.

The base space is an m-dimensional manifold $\mathcal{M}$. It is common to call k the *rank* of fibre bundle (or fibre dimension), denoted by dim E, although the total space E is $(m+k)$-dimensional. The transition functions belong to the structure group $GL(k, \mathbb{F})$ since it maps a vector space onto another vector space of the same dimension isomorphically, i.e. $f_{ij} : U_i \cap U_j \to \mathrm{Hom}(V, V)$ such that $(p_i, v_i) = (p_i, f_{ij} v_j)$ for $p_i \in \mathcal{M}$, $v_i \in V$.

There are many natural examples of vector bundles such as tangent bundle, cotangent bundle (dual to tangent bundle) of a manifold and their tensor products.

Definition 12. A **tangent bundle** $T\mathcal{M}$ is a disjoint union of the tangent spaces at all points of $\mathcal{M}$: $T\mathcal{M} = \bigsqcup_{p \in \mathcal{M}} T_p\mathcal{M}$.

The tangent bundle is a smooth vector bundle of rank k over a k-dimensional manifold $\mathcal{M}$. Here $\mathcal{M}$ is the base of this bundle, and the $2k$-dimensional manifold $T\mathcal{M}$ itself is its total space. Its fibre is $\mathbb{F}^k$ and the structure group is $GL(k, \mathbb{F})$. There is a natural projection map $\pi : T\mathcal{M} \to \mathcal{M}$ which, for each $x \in \mathcal{M}$, sends every vector $X \in T\mathcal{M}$ to x. The preimages $\pi^{-1}(x) \to T_x\mathcal{M}$ are the fibres of this bundle. Sections of $T\mathcal{M}$ are the vector fields $\chi(\mathcal{M})$ on $\mathcal{M}$, i.e. $\Gamma(\mathcal{M}, T\mathcal{M}) = \chi(\mathcal{M})$.

A vector bundle whose fibre is one-dimensional ($F = \mathbb{R}$ or $\mathbb{C}$) is called a **line bundle**. The structure group $GL(1, \mathbb{R}) = \mathbb{R} - \{0\}$ or $GL(1, \mathbb{C}) = \mathbb{C} - \{0\}$ is abelian. A cylinder $\mathbf{S}^1 \times \mathbb{R}$ is a trivial $\mathbb{R}$-line bundle. A Möbius strip is also a $\mathbb{R}$-line bundle. One of the applications of vector bundles is seen in the *(trivial) complex line bundle* $L = \mathbb{R}^3 \times \mathbb{C}$ which is associated with the non-relativistic quantum mechanics defined on $\mathbb{R}^3$. The wavefunction $\psi(x)$ is simply a section of L.

Principal Bundle

Definition 13. A smooth **principal bundle** $P \xrightarrow{\pi} \mathcal{M}$ is a smooth fibre bundle whose fibre F is diffeomorphic to a Lie group G, such that the structure group reduces to G, acting on itself by left multiplication. We often call it a **principal G-bundle** over $\mathcal{M}$ and denote it by $P(\mathcal{M}, G)$.

It is worth noting that:

1. For each $p \in \mathcal{M}$, F_p is diffeomorphic to a Lie group G, but not in a canonical way – in particular, there is no special element $e \in F_p$ that we can call the identity.

2. Having reduced the structure group to G, we have chosen a special set of local trivialisations $\varphi_i : \pi^{-1}(U_i) \to U_i \times G$ that are related to each other on each fibre by the left action of G. This has the effect of inducing a natural fibre preserving *right action*: $E \times G \to E$ $((p, g) \mapsto pg)$, defined in terms of any local trivialisation $\varphi_i(p) = (\pi(p), f_i(p)) \in U_i \times G$ by $f_i(pg) = f_i(p)g$. This is independent of the choice because the right action of G on itself commutes with the left action of the transition functions. Indeed, if $p \in F_p$ for $p \in U_i \cap U_j$, then $f_j(pg) = f_{ji}(p)f_i(pg) = f_{ji}(p)f_i(p)g = f_j(p)g$.

This right action of G is both

1. free, i.e. without fixed points, meaning $pg = p$ for some $p \in E$ and $g \in G$ if and only if g is the identity, and

2. transitive, i.e. for any $p, q \in F_p$, there exists $g \in G$ such that $q = pg$.

Therefore, we have an equivalent definition for principal bundles.

Definition 14. A smooth principal bundle is a smooth fibre bundle $P \xrightarrow{\pi} \mathcal{M}$ together with a Lie group G and a fibre preserving right action $E \times G \to E$ which restricts to each fibre freely and transitively.

For principal bundles there is a one-to-one correspondence between sections and trivialisations. A principal bundle P is trivial ($P = \mathcal{M} \times G$) if and only if it admits a smooth global section $s : \mathcal{M} \to P$.

Associated Bundle

If two fibre bundles E and E', with the same base space $\mathcal{M}$ and structure group G, also share the same trivialising neighbourhoods U and transition functions f_{ij}, then they are each called an **associated bundle** with regard to the other. It is possible to construct (up to isomorphism) a unique principal G-bundle associated to a given fibre bundle and *vice-versa*.

In particular, if we are given a principal G-bundle $P(\mathcal{M}, G)$ over $\mathcal{M}$ and a left action of G on a fibre F, $\rho : G \times F \to F$, then the fibre bundle E associated to P is,

$$E := P \times_G F = P \times F/G, \tag{2.5}$$

where the quotient space collapses all points in the product space $P \times F$ which are related by the right action of some $g \in G$ on P and the right action of g^{-1} on F. The associated bundle E is hence defined with an equivalence class in which $(p, f) \sim (p \cdot g, \rho(g^{-1}, f))$, for all $(p, f) \in (P, F)$ and all $g \in G$.

Frame Bundle

Moreover, we can define an associated bundle called the *frame bundle*.

Definition 15. A **(local) frame** over an open set $U \subset \mathcal{M}$ is an ordered set $(e_1, \ldots, e_k)$ of smooth sections $e_i \in \Gamma(U, E)$ which span each fibre F_p for $p \in U$.

Equivalently, a local frame can be viewed as a linear isomorphism from U to F_p. Restricting to a single fibre, a frame on F_p is simply an ordered basis of F_p. The set of all frames is hence the fibre of $F(E)$ over p. Here the general linear group $GL(k, \mathbb{F})$ acts naturally on $F(E)$ via a change of basis, giving the frame bundle the structure of a principal $GL(k, \mathbb{F})$-bundle.

A **global frame** is then a frame over the entire base space $\mathcal{M}$ which only exists if the bundle is trivialisable. This follows from the result that a frame $(e_1, \ldots, e_k)$ over $U \subset \mathcal{M}$ for the bundle E determines a trivialisation $\varphi : E|_U \to U \times \mathbb{F}^k$ such that for each $p \in U$ and $i \in \{1, \ldots, k\}$, $\varphi(e_i(p)) = (p, v_i)$ where $(v_1, \ldots, v_k)$ is the standard basis of unit vectors in $\mathbb{F}^k$.

Now we denote by $F(F_p)$ the set of all frames on F_p, and let $F(E) = \bigcup_{p \in \mathcal{M}} F(F_p)$. This space has a natural topology and smooth manifold structure such that any frame over $U \subset \mathcal{M}$ defines a smooth map $U \to F(E)$. With a smooth projection map $\pi : F(E) \to \mathcal{M}$, we define a *frame bundle*.

Definition 16. Given a vector bundle E of rank k whose fibre is $\mathbb{F}^k$ (can be $\mathbb{R}^k$ or $\mathbb{C}^k$), the **frame bundle** of E is the principal $GL(k, \mathbb{F})$-bundle associated to E, and is denoted $F(E) \equiv (F(E), \pi, \mathcal{M}, \mathbb{F}^k, GL(k, \mathbb{F}))$.

2.4 Connection, Curvature, and Chern Classes

The notion of connection generalises the concept of directional derivative. We knew that the derivative d/dx acts on the space of functions. For a vector bundle E, the space of functions is generalised to the space of sections $\Gamma(E) = \{s : \mathcal{M} \to E | \pi \circ s = id_{\mathcal{M}}\}$, as introduced before. Then a **connection** is defined as a linear operator, $D : \Gamma(E) \to \Gamma(E \otimes T^*\mathcal{M})$, satisfying the Leibniz rule $D(fs) = df \otimes s + fDs$, for $f \in C^\infty(\mathcal{M})$, $s \in \Gamma(E)$. Here d is the usual exterior derivative. Connections allow us to determine how sections of the bundle vary as we go from fibre to fibre. This is known as a **parallel transport** in which we should determine a local expression for the connection.

In local coordinates, let $\{e_i\}$ be a basis of sections (i.e. a frame) so that every section s can be represented by $s = \sum_i s_i e_i$. We have $De_i = \sum_j \theta_{ij} e_j$, where the connection matrix $A = (\theta_{ij})$ is represented by a matrix of one-forms. For a different set of local coordinates related by a gauge transformation $g : \mathcal{M} \to G \subset \mathrm{Hom}(V, V)$, the connection A is then represented by $A' = gAg^{-1} + dg\, g^{-1}$, where $dg\, g^{-1}$ is the Maurer-Cartan form of G. For each connection A, we define the **covariant derivative** as $D_A = d + A$. This covariant derivative transforms covariantly under a gauge transformation: $D_{A'} = gD_Ag^{-1}$.

Curvature of a connection A is defined as $\Omega = D_A^2 = dA + A \wedge A$, which enjoys two important properties: Bianchi identity, $D\Omega = 0$, and covariant transformation $\Omega' = g\Omega g^{-1}$ for a different set of local coordinates differed by a gauge transformation g. In Chern-Simons (CS) theory, the curvature is the field strength $F \equiv dA + A \wedge A$.

One may construct many fibre bundles over the base space $\mathcal{M}$, depending on the choice of the transition functions. Some natural questions one may ask are how many bundles there are over $\mathcal{M}$ with given F and G, and how much they differ from a trivial bundle $\mathcal{M} \times F$. One brilliant way to classify the fibre bundles is using **characteristic classes**, which are subsets of the cohomology classes of $\mathcal{M}$ that measure the non-triviality or twisting of a bundle. In this sense, they are obstructions which prevent a bundle from being a trivial bundle.

One particular essential characteristic class is that associated with complex vector bundles E, namely the *Chern classes*. From the Bianchi identity, we can verify that the i-th **Chern classes** of E, $c_i(E) = \frac{1}{2\pi i} \operatorname{Tr} \Omega^i$ is a closed form in $H^{2i}(\mathcal{M}, \mathbb{R})$. A representative of each Chern class, called a **Chern form**, $c_i(E)$ of E is given by the coefficient of the characteristic polynomial of the curvature form Ω of E, $\det\left(I - \frac{1}{2\pi i}\lambda\Omega\right) = \Sigma_i c_i\left(E\right)\lambda^i$. If we modify A to $A + \delta A$, the curvature is changed to $\Omega + D_A \delta A$. Also, $\operatorname{Tr} D\delta A = d \operatorname{Tr} \delta A$. For any 2-cycle Σ, we have

$$\int_\Sigma \frac{1}{2\pi i} \operatorname{Tr} \Omega' = \int_\Sigma \frac{1}{2\pi i} \operatorname{Tr}(\Omega + D_A \delta A) = \int_\Sigma \frac{1}{2\pi i} \operatorname{Tr} \Omega. \tag{2.6}$$

By the same reasoning plus the Bianchi identity, the second Chern class $c_2(E) = \frac{1}{2\pi i} \operatorname{Tr}(\Omega \wedge \Omega)$ is also a topological invariant. One interesting fact is that the CS action S of the boundary $\partial\mathcal{M}$ and the second Chern class $c_2(E)$ of $\mathcal{M}$ are related by $dS = c_2(E)$, meaning that the secondary characteristic class is defined by the CS 3-form which trivialises the difference between two curvature forms. Further, by the same approach, it is easy to verify that the Chern classes $c_i(E)$ as cohomology classes are independent of choices of A. This is the beautiful **Chern-Weil theory**.

3 Knots and Links

In the mathematical sense, a knot is a possibly tangled loop, freely floating in ordinary space. It is a remarkably complicated thing, even with all the sophisticated techniques of modern topology, it has resisted a definitive treatment.

Knots have a fascinating connection with quantum physics. In fact, knot theory, which aims to classify all knots, was largely motivated by physics of the late eighteenth century. Lord Kelvin (Sir William Thomson) conjectured that atoms were vortices of aether swirling along knotted paths in space in the 1860s[1]. This led to the classification of knots by Peter Tait, in 1867, according to their number of *crossings* when drawn on a plane. About twenty years later, he published a table of knots with up to ten crossings along with the Tait conjectures. In this section, we state some general ideas of knots and links and their invariants; a detailed analysis of knot theory can be found in [15–18]. We begin with definitions of knots and links.

3.1 Definitions

Definition 17. A **knot** $\mathcal{K}$ is a smooth embedding of codimension 2 in a 3-manifold $\mathcal{M}$, that is diffeomorphic to $\mathbf{S}^1$. Formally, $\mathcal{K}$ is a *simple closed curve* (also known as *Jordan curve*) that is a nearly injective and continuous function $\mathcal{K} : [0,1] \to \mathcal{M}$, with the only "non-injectivity" being $\mathcal{K}(0) = \mathcal{K}(1)$.

The most common 3-manifolds are $\mathbb{R}^3$ or $\mathbf{S}^3$.

Definition 18. A **link** $\mathcal{L}$ is a smooth embedding into a 3-manifold $\mathcal{M}$ that is diffeomorphic to a finite disjoint union of knots: $\mathcal{L} = \mathcal{K}_1 \sqcup \mathcal{K}_2 \sqcup \cdots \sqcup \mathcal{K}_n$.

Each knot $\mathcal{K}_i$ is called a complement of the link. The number of complements of a link $\mathcal{L}$ is called the multiplicity $\mu(\mathcal{L})$ of the link. A subset of the components of the link is called a sublink. From now on, the discussion focusses on links whereby all the notions on links apply similarly to knots.

Definition 19. A **link diagram** D is the projection of a link onto a plane with crossings indicated. A **crossing** is a point on the diagram D of the link where the link passes over or under itself. A **crossing number** of a link $\mathcal{L}$, denoted by $c(\mathcal{L})$, is the minimum number of crossings over the set of diagrams D of $\mathcal{L}$,

$$c(\mathcal{L}) = \min\{c(D) \,|\, D \in \text{diagrams of } \mathcal{L}\}. \tag{3.1}$$

[1] Lord Kelvin's conjecture implies that the more complex the knot, the heavier the atom. The conjecture was ultimately fruitless, as we know well today. On the other hand, other attempts to find a connection between the topological properties of knots and topological properties in certain electromagnetic phenomena proved successful [15].

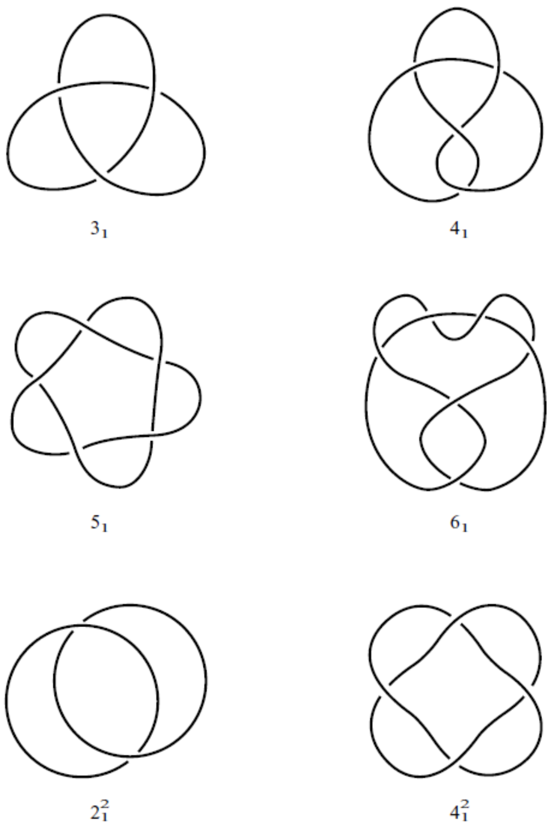


Figure 2: Some examples of knot diagrams and link diagrams.

A standard notation for knots and links is x_n^L: x indicates the crossing number (i.e. $c(\mathcal{L})$), L the number of components (only for links with $L > 1$) and n is a number used to enumerate knots and links in a given set characterized by x and L. The simplest example is the *trivial knot* or *unknot*, 0_1, which is just the circle, $C = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1,\ z = 0\}$. More interesting examples are depicted in Fig. 2. The knot 3_1 is known as the *trefoil knot*, whereas 4_1 is known as the *figure-eight knot*. The link 2_1^2 is called the *Hopf link*.

Remark. Since a link is a manifold (a submanifold of $\mathcal{M}$), it makes sense to give it an orientation, hence a link with an orientation is called an **oriented link**.

Ambient Isotopy

As one would expect, two links are said to be equivalent or **homotopic** to each other if they can be continuously deformed into one another by bending, shrinking and expanding operations, without breaking them. However, in knot theory, a stronger definition than homotopy is required to determine whether two embeddings are connected. It is the *ambient isotopy* which we now define.

Definition 20. An **isotopy** is a homotopy for which $H(s,t)$ is injective for all $t \in I = [0,1]$.

The injectivity removes the possibility of a knot passing through itself. However, this is still not enough because the knots have no thickness (infinitesimally thin), we can deform a knot by tightening it to the point where it disappears. The result is that all knots are isotopic to the unknot. So, we need the concept of *ambient isotopy* which allows us to stretch, compress and distort the whole ambient space containing the knot, instead of just the knot. The term "ambient" comes from ambient space, which is the space surrounding a mathematical object along with the object itself. Some common ambient spaces are vector spaces, affine spaces, projective spaces, and Grassmann spaces.

Definition 21. Two links $\mathcal{L}$ and $\mathcal{L}'$ are **ambient isotopic** if there is a smooth map $\alpha : [0,1] \times \mathcal{M} \to \mathcal{M}$ such that for each value $t \in [0,1]$, the map $\alpha(t,\cdot) : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, and $\alpha(0,\cdot)$ is the identity map on $\mathcal{M}$, while $\alpha(1,\cdot)$ maps $\mathcal{L}$ to $\mathcal{L}'$.

All links are classified up to ambient isotopy. Naturally enough, an equivalence class of links with ambient isotopy is called an **ambient isotopy class**. This defines the link diagram.

Reidemeister Moves

There is now a basic question: how do we know two different diagrams represent the same ambient isotopy class, i.e. they are ambient isotopic? This question has a beautiful answer. Two diagrams D_1 and D_2 with the associated links $\mathcal{L}_1$ and $\mathcal{L}_2$ are ambient isotopic if and only if a finite sequence of **Reidemeister moves** (Fig. 3) which transforms D_1 to D_2 exists. These moves are local changes to a diagram and are very important in knot theory as they encode the symmetry structure in the link classification problem. Indeed we can construct link invariants of ambient isotopy by finding invariants of the symmetry group generated by the Reidemeister moves.

The Reidemeister moves can be used to determine which knots are ambient isotopic to their mirror images and which are not. For example, in Fig. 4, the trefoil is not isotopic to its mirror image, while the figure-eight knot is. Such knots which are isotopic to their mirror images are said to be **amphichiral**.

3.2 Invariants

Two useful quantities to distinguish an oriented link diagram D from the other one is the *writhe* and the *linking number*.

Definition 22. A **writhe** (**self-linking number**, **cotorsion** or **Tait number**) of D, is defined as $\phi(D) = \sum_{p \in D} \epsilon(p)$, which is the sum over all crossing points p in D with $\epsilon(p) = \pm 1$ being a sign associated to the crossings as indicated in Fig. 5.

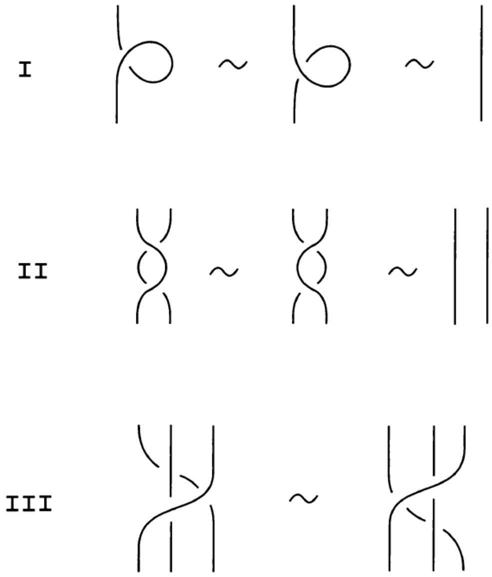
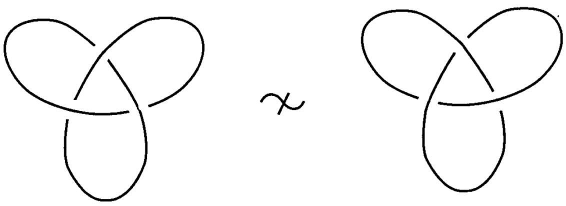
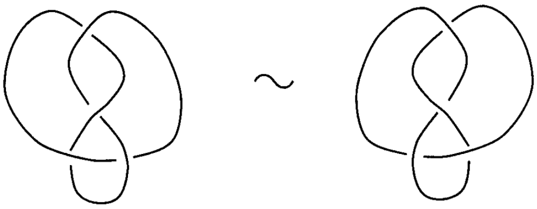


Figure 3: Reidemeister moves.



(a)



(b)

Figure 4: (a) The trefoil is not ambient isotopic to its mirror image, (b) the figure-eight knot is ambient isotopic to its mirror image.

Equivalently, the writhe can be written as $\phi(D) = n_+(D) - n_-(D)$ for $n_\pm(D)$ the number of positive (negative) crossings. The problem is that it is not an ambient isotopy invariant[2].

We now define an ambient isotopy invariant for any two linked oriented knots

[2]The writhe is only a *regular isotopy* invariant, which means that it does not change under a sequence of Reidemeister moves of types II and III.

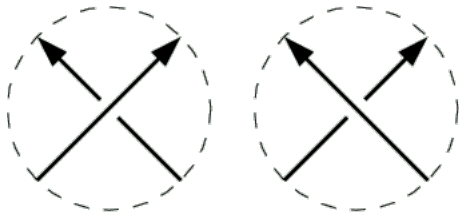


Figure 5: Positive crossing (or over crossing) is assigned with a sign +1 and negative crossing (under crossing) a sign −1 when computing the linking number of two knots.

Figure 6: Two different orientations of trefoil knot.

$\mathcal{K}_1, \mathcal{K}_2$, known as the **linking number**:

$$lk(\mathcal{K}_1, \mathcal{K}_2) = \frac{1}{2} \sum_p \epsilon(p), \tag{3.2}$$

For example, once an orientation is chosen for the trefoil knot shown in Fig. 6, one finds two inequivalent oriented links with linking numbers ±1. The linking number of a link $\mathcal{L}$ with components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$ is defined as:

$$lk(\mathcal{L}) = \sum_{\alpha<\beta} lk(\mathcal{K}_\alpha, \mathcal{K}_\beta). \tag{3.3}$$

3.3 Framings

In the previous section 3.2, we stated that the writhe which defines the self-linking number of a link is not invariant under the deformations of the link and its ambient space. In order to make sense of the self-linking of a link $\mathcal{L}$ topologically, one needs the idea of *framing* of a link. Just as an orientation may be represented by a nonvanishing vector field tangent to the link, a *framing* of a link is a vector field that is *nowhere* tangent to the link. Note that for any link $\mathcal{L}$ embedded in $\mathcal{M}$, the tangent space of the link, at any point $p \in \mathcal{L}$, is a subspace of the tangent space of $\mathcal{M}$,

$$T_p\mathcal{L} \subseteq T_p\mathcal{M} \cong \mathcal{M}. \tag{3.4}$$

We call a smooth function from $\mathcal{L}$ to $\mathcal{M}$ a vector field on $\mathcal{L}$, being careful that it is not necessarily a tangent vector field.

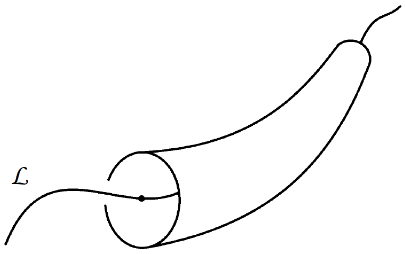


Figure 7: Tubular neighbourhood of a link $\mathcal{L}$.

Definition 23. The **framing** of a link $\mathcal{L} \subset \mathcal{M}$ is a normal vector field n on $\mathcal{L}$, such that $n_p \notin T_p\mathcal{L}$ for all $p \in \mathcal{L}$, as in Fig. 8a. A link equipped with a framing is called a **framed link**, $\mathcal{L}_f$. The **standard framing**, also called the **canonical framing** or **blackboard framing**, is the unit vector field that is everywhere orthogonal to the plane of projection of the link, oriented towards the point of projection.

The idea is that by displacing $\mathcal{L}$ slightly in the direction of the vector field v one gets a new link $\mathcal{L}_f$, in the tubular neighbourhood of $\mathcal{L}$. The tubular neighbourhood $\mathcal{L}$ is simply a torus whose core is $\mathcal{L}$, as in Fig. 7.

We may think of the framing as a thickening of the link into a tiny ribbon bounded by $\mathcal{L}$ and $\mathcal{L}_f$, which is drawn in Fig. 8a. It is clear that the self-linking number defined this way depends not on the actual vector field used to displace $\mathcal{L}$ to $\mathcal{L}_f$ but only on the *topological class* of this vector field; and indeed by a framing we mean only the topological class. The projective diagram of this link will then have crossings between the link and its framing. These crossing correspond to the *twists* in the ribbon itself. We mentioned that a normal knot has no thickness, thus it cannot have any twisting.

Cotorsion (Writhe / Self-linking Number)

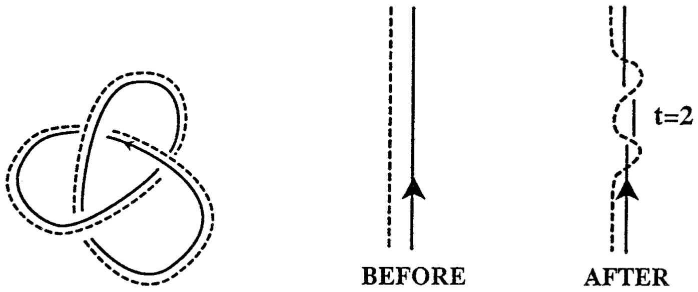
Through a convenient framing of $\mathcal{L}$, the self-linking number or cotorsion, which is the linking number of a link and its framing, $lk(\mathcal{L}, \mathcal{L}_f)$, is then a t**-fold twist** in the framing of $\mathcal{L}$:

$$\phi(\mathcal{L}) = lk(\mathcal{L}, \mathcal{L}_f) = t(\mathcal{L}); \tag{3.5}$$

as illustrated in Fig. 8b. This provides us a way to obtain the linking number. We simply need to look at the link diagram and calculate the number of crossing points (for different links) and twists (for the same link).

Another way is a more direct geometrical interpretation of the linking number. By expressing a linking number in terms of the Gauss integral,

$$lk(\mathcal{L}_\alpha, \mathcal{L}_\beta) = \frac{1}{4\pi} \oint_{\mathcal{L}_\alpha} dx^\mu \oint_{\mathcal{L}_\beta} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}, \tag{3.6}$$



(a) (b)

Figure 8: (a) Framing of a knot. (b) The framing is shifted by 2 units by making a 2-fold twist.

Figure 9: Two framings of the unknot.

where the distance $|x-y|$ is computed by means of the flat (Euclidean) metric of $\mathcal{M}$. For $\alpha = \beta$, the integral is

$$\tilde{\phi}(\mathcal{L}) = \frac{1}{4\pi}\oint_{\mathcal{L}} dx^\mu \oint_{\mathcal{L}} dy^\nu \epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}. \tag{3.7}$$

This integral is well-defined and finite except near $x = y$. It is also not an ambient isotopy invariant. By means of framing, we can remedy this integral. Having another link $\mathcal{L}_f$, the integral becomes the cotorsion $\phi(\mathcal{L})$:

$$\phi(\mathcal{L}) = \frac{1}{4\pi}\oint_{\mathcal{L}} dx^\mu \oint_{\mathcal{L}_f} dy^\nu \epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3} = lk(\mathcal{L},\mathcal{L}_f). \tag{3.8}$$

One thing to notice here is that the integrand in (3.8) is dependent on the metric of $\mathcal{M}$, but the result of the integral is metric-independent, i.e. it depends exclusively on topology.

Two framed links, $\mathcal{L}_{f1}$ and $\mathcal{L}_{f2}$ are equivalent, if there exists an ambient isotopy that takes the link $\mathcal{L}_1$ to $\mathcal{L}_2$ and also takes $\mathcal{L}_{f1}$ to $\mathcal{L}_{f2}$. An example depicted in Fig. 9 shows that the two framings of the unknot are not isotopic. To use a link invariant to assess whether two links are equivalent, one has to make sure that they are both in the same framing, because link invariants are not guaranteed to be constant over a change of framing.

Canonical Framing

Now, is it always possible to pick a standard (or canonical) framing such that the cotorsion vanishes: $lk(\mathcal{L}, \mathcal{L}_f) = 0$, so that we can hide the self-linking question? On $\mathbf{S}^3$, there is a canonical framing of every knot and link. On general three manifolds, this cannot be done since the cotorsion may be ill-defined or may differ from an integer by a definite fraction. Even when the canonical framing does exist, it is not convenient to be restricted to using it, since natural operations (like the *surgery* in Section 5.1) may not preserve it. We will later see that framings arise naturally when considering regularised Wilson loops in a TQFT (see Section 5.2).

3.4 Polynomial Invariants

One of the earliest knot polynomial discovered was the Alexander polynomial by James W. Alexander in 1923 [19], who was a pioneer of algebraic topology. Its significance was not realised until the discovery of the Jones polynomial by Vaughan Jones in 1984 [20]. This discovery led to a flood of new surprises that is continuing to this very day. Jones polynomial is a one-variable polynomial assigned to a knot or a link in the manifold $\mathcal{M}$. To define it, we require some terminologies.

Definition 24. The **Laurent polynomial** over a field $\mathbb{F}$ is defined as,

$$p = \sum_{k\in\mathbb{Z}} p_k t^k \quad p_k \in \mathbb{F}, \tag{3.9}$$

where t is a variable and p_k is nonzero for finitely many k.

Definition 25. The **Kauffman bracket** $\langle \cdot \rangle$ is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in a variable A. It maps a diagram D to $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$ and is characterised by,

1. $\langle \bigcirc \rangle = 1$,
2. $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2)\langle D \rangle$,
3. $\langle D_+ \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle$.

Note that as a consequence of this we have $\langle D_- \rangle = A^{-1}\langle D_0 \rangle + A\langle D_\infty \rangle$. Here, $\bigcirc$ is the unknot. The notations D_+, D_-, D_0, D_∞ refer to any link diagrams D, which have been modified in some neighbourhood of a point $p \in D$, as shown in Fig. 10, respectively.

Definition 26. The **skein relations** are formulae that define the link invariants by saying how a link invariant is affected by modifying some small portion of the link diagram while keeping the rest of the diagram fixed.

We now define the Jones polynomial as follows.

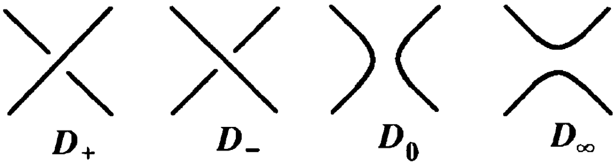


Figure 10: Locally modified link diagrams D_+, D_-, D_0, D_∞, which are collectively known as the unoriented Conway quadruple.

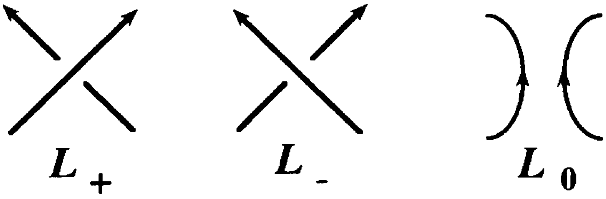


Figure 11: Oriented Conway triple.

Definition 27. The **Jones polynomial** of an oriented link $\mathcal{L}$, denoted by $V(\mathcal{L})$ or $V^{\mathcal{L}}(t)$, is the Laurent polynomial in $t^{1/2}$ with integer coefficients, defined by

$$V^{\mathcal{L}}(t) = (-A)^{-3\phi(D)}\langle D\rangle\Big|_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2},\, t^{1/2}], \tag{3.10}$$

where D is a diagram of oriented link $\mathcal{L}$ and $\phi(D)$ is the cotorsion of D.

There is another proposition often used as an alternative definition of the Jones polynomial.

Proposition. The **Jones polynomial** is a function,

$$V : \{\text{oriented links in } \mathcal{M}\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}], \tag{3.11}$$

which is defined by the axioms:

1. Invariance: $V(\mathcal{L})$ is invariant under ambient isotopy of $\mathcal{L}$, i.e. if $\mathcal{L}_1$ and $\mathcal{L}_2$ are ambient isotopic, then $V(\mathcal{L}_1) = V(\mathcal{L}_2)$.

2. Normalisation: $V(\bigcirc) = 1$, where $\bigcirc$ denotes the unknot or trivial knot.

3. Skein relation: Whenever three oriented links $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{L}_0$ are the same except in the neighbourhood of a point where they differ as in the (oriented) Conway triple (Fig. 11), then we have the skein relation,

$$t^{-1}V(\mathcal{L}_+) - tV(\mathcal{L}_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0. \tag{3.12}$$

According to definitions of Jones polynomial, there are two ways to compute the Jones polynomial of a given link or knot, namely, using the Kauffman brackets and the skein relation. The relatively more popular (and straightforward) method is the skein relation, which we shall illustrate using the Hopf links.

Example. *We consider the positive and negative Hopf links with their variants, as in Fig. 12. Using the skein relation (3.12), we have*

$$\begin{aligned} t^{-1}V(H_+) - tV(\mathcal{L}_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0, \\ t^{-1}V(\mathcal{L}_+) - tV(H_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0, \end{aligned} \tag{3.13}$$

for the positive Hopf link ($H_+ \equiv \mathcal{L}_+$) and the negative Hopf link ($H_- \equiv \mathcal{L}_-$), respectively. For $\mathcal{L}_0$ in both cases, they are just unknot. By definition, $V(\mathcal{L}_0) = 1$. The $\mathcal{L}_-$ in the former case and the $\mathcal{L}_+$ in the latter case are both two unlinked unknots. To evaluate the Jones polynomial of the two unlinked unknots, we consider its Skein triple (Fig. 13) and apply the skein relation,

$$t^{-1}V(\mathcal{L}_+) - tV(\mathcal{L}_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0. \tag{3.14}$$

Since both $\mathcal{L}_+$ and $\mathcal{L}_-$ are unknots, $V(\mathcal{L}_+) = V(\mathcal{L}_-) = 1$, then

$$V(\mathcal{L}_0) = -(t^{1/2} + t^{-1/2}). \tag{3.15}$$

Returning to evaluating the Jones polynomial of Hopf links, we have

$$\begin{aligned} t^{-1}V(H_+) + t(t^{1/2} + t^{-1/2}) + (t^{-1/2} - t^{1/2}) = 0, \\ -t^{-1}(t^{1/2} + t^{-1/2}) - tV(H_-) + (t^{-1/2} - t^{1/2}) = 0, \end{aligned} \tag{3.16}$$

which give us

$$V(H_+) = -t^{1/2}(1 + t^2) \quad \text{and} \quad V(H_-) = -t^{-1/2}(1 + t^{-2}). \tag{3.17}$$

Since the Jones polynomial $V(H_+)$ is not invariant under $t \to t^{-1}$, we deduce that the positive Hopf link is not equivalent to its mirror image; same deduction applies to the negative Hopf link.

There is a generalisation of the Jones polynomial, known as the HOMFLY polynomial, which was introduced by Freyd *et al.* (1985) [21]. It depends on two variables q and λ. Some texts refer to it as HOMFLYPT polynomial in the recognition of independent work carried out by Józef H. Przytycki and Paweł Traczyk.

Definition 28. The **HOMFLY polynomial**, $P(\mathcal{L})$ or $P^{\mathcal{L}}(q, \lambda)$, of an oriented link $\mathcal{L}$ is defined by the following three axioms:

1. Invariance: $P^{\mathcal{L}}(q, \lambda)$ is invariant under ambient isotopy of $\mathcal{L}$.
2. Normalisation: $P(\bigcirc) = 1$.
3. Skein relation: $q^{-1}P(\mathcal{L}_+) - qP(\mathcal{L}_-) = \lambda P(\mathcal{L}_0)$.

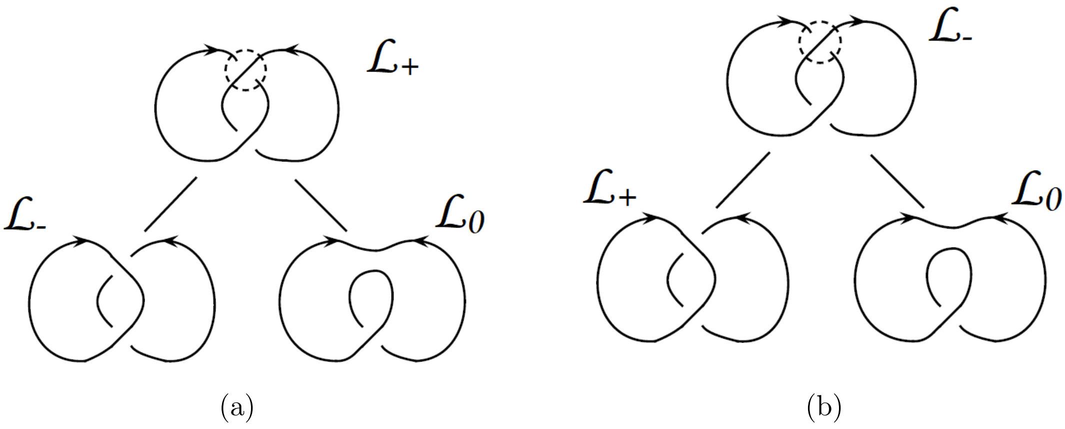


Figure 12: *Skein trees* for the (a) positive Hopf link ($H_+ \equiv \mathcal{L}_+$ with positive linking number) and (b) negative Hopf link ($H_- \equiv \mathcal{L}_-$ with negative linking number).

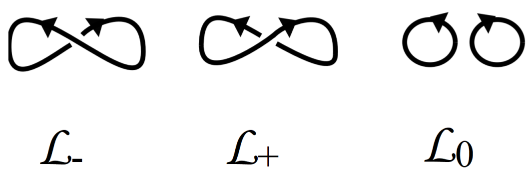


Figure 13: The Skein triple for two unlinked unknots ($\mathcal{L}_0$).

The Jones polynomial and its relatives turn out to be closely related to the correlation function (4.15) (or the vacuum expectation value of products of Wilson loops) with a gauge group G being considered. One will obtain the *HOMFLY polynomial* from the Chern-Simons (CS) theory when $G = SU(N)$ or $U(N)$[3] and all the Wilson loop components are taken in the fundamental representation[4] $\mathcal{R}_\alpha = \square$. Explicitly, we have

$$W_{\square\cdots\square}(\mathcal{L}) = \lambda^{lk(\mathcal{L})}\left(\frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right) P^{\mathcal{L}}(q,\lambda), \tag{3.18}$$

where $lk(\mathcal{L})$ is the linking number of $\mathcal{L}$, and the variables q and λ are related to the CS variables k and $h^\vee$ as

$$q = \exp\left(\frac{2\pi i}{k+h^\vee}\right), \quad \lambda = q^{h^\vee}, \tag{3.19}$$

[3]We work in large N limit. Our results for HOMFLY polynomial are not sensitive to this choice.

[4]The irreducible representations of $SU(N)$ are labelled by highest weights or equivalently by the lengths of rows in a Young tableau, l_i, where $l_1 \geq l_2 \geq \ldots$.

where k is the level of CS theory and $h^\vee$ is the dual Coxeter number[5] of the gauge group G ($h^\vee = N$ for $G = SU(N)$). For $N = 2$, the HOMFLY polynomial reduces to the one-variable polynomial, the *Jones polynomial*, $V^{\mathcal{K}}(t)$, provided the identification $t = \exp(\frac{2\pi i}{k+2})$. The case of $SU(2)$ as gauge group and Wilson loops carrying a higher dimensional representation (of spin $s/2$, $s \in \mathbb{Z}_+$) leads to the *Akutsu-Wadati polynomials* [22]. When the gauge group of CS theory is $SO(N)$, $W_{\Box\cdots\Box}(\mathcal{L})$ is closely related to the *Kauffman polynomial* [23].

[5]The dual Coxeter labels $a_i^\vee$ ($i = 1, \ldots, r = \mathrm{rank}(\mathfrak{g})$) of a Lie algebra $\mathfrak{g}$ are the expansion coefficients of highest coroot $\theta^\vee$ in terms of the coroot $\alpha^{\vee(i)}$: $\theta^\vee = \sum_{i=1}^r a_i^\vee \alpha^{\vee(i)}$. The dual Coxeter number $h^\vee$ is defined as $h^\vee = 1 + \sum_i^r a_i^\vee$. It can also be defined by the quadratic Casimir of the adjoint representation of $\mathfrak{g}$: $f^{ab}{}_c f^{bc}{}_d = 2h^\vee \delta^{ad}$, where $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$.

4 Chern-Simons Theory

The intervention of topology into QFT was first recognised in the discovery of magnetic monopoles in $SU(2)$ gauge theory with scalar field-Georgi-Glashow model [24, 25]. The magnetic charge is of topological nature and simple topological considerations were used to prove the existence of magnetic monopoles in a large class of gauge theories, including all grand unification theories. QFT where topological invariance is manifest, is then coined the term "Topological Quantum Field Theory" or TQFT in short.

There are two types of TQFTs, according to the terminology of Birmingham *et al.* [26]. These are called topological field theories of Schwarz type and Witten type. In the Schwarz type theories, the classical action is invariant under a change of background metric and this symmetry is preserved upon quantisation. The Witten type is a cohomological field theory with a very different flavour, which we will not discuss in this review (a detailed discussion can be found in [4, 27]).

4.1 The Chern-Simons Action

Chern–Simons (CS) theory is a general covariant 3-dimensional Schwarz type TQFT with a gauge invariant Lagrangian that does *not* contain a metric. The CS theory can be defined on any topological 3-manifold $\mathcal{M}$, with or without boundary. We begin on a compact, oriented three-manifold $\mathcal{M}$ with a *compact simple* gauge group G. Since CS theory is a gauge theory, its classical configuration on $\mathcal{M}$ with gauge group G is described by a principal G-bundle E over $\mathcal{M}$, which we may choose to be topologically trivial (a technical convenience). The G-gauge connection of this G-trivial bundle E is characterised by a gauge field A_i^a, which is a connection one-form valued in the Lie algebra $\mathfrak{g}$ of the Lie group G. Here a runs over a basis of $\mathfrak{g}$, and i is tangent to $\mathcal{M}$.

In general, the connection A is defined locally on individual coordinate patches, and the values of A on different patches are related by maps known as *gauge transformations*. An infinitesimal gauge transformation is

$$A_i \to A_i - D_i\varepsilon, \tag{4.1}$$

where ε, a generator of the gauge group G, is a $\mathfrak{g}$-valued zero form. These are characterised by the assertion that the covariant derivative,

$$D_i\varepsilon = \partial_i\varepsilon + [A_i, \varepsilon], \tag{4.2}$$

transforms in the adjoint representation of the Lie group G. The curvature (or field strength), defined as the "square" of the covariant derivative with itself, is the $\mathfrak{g}$-valued two form

$$F_{ij} = [D_i, D_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \tag{4.3}$$

It also transforms in the adjoint representation.

The CS action is an integral of the CS three-form on a compact oriented 3-manifold,

$$\begin{aligned} S &= \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\,(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \\ &= \frac{k}{4\pi}\int d^3x\,\epsilon^{ijk}\ \mathrm{Tr}\,(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k), \end{aligned} \tag{4.4}$$

where k is the (inverse) coupling constant and is known as the **level** of the CS theory. The CS theory is weakly coupled as $k\to\infty$ and becomes strongly coupled as k gets smaller. The trace Tr denotes an invariant, bilinear, quadratic form on the Lie algebra $\mathfrak{g}$ and is a multiple of the Killing form. For instance, for $G = SU(N)$, the trace Tr is given concretely in the fundamental representation of $SU(N)$.

Since the CS action (4.4) does not involve the metric, the resulting theory is topological. This explains why the standard Yang-Mills action

$$S_{\mathrm{YM}} = \int_{\mathcal{M}} \sqrt{g}g^{ik}g^{jl}\ \mathrm{Tr}\,(F_{ij}F_{kl}), \tag{4.5}$$

is not chosen, as it depends on the choice of a metric g_{ij} of $\mathcal{M}$.

This CS action is nonabelian and has been studied classically by Zuckerman [28]. The abelian CS theory, where its action is only the first term on the right-hand side of (4.4), $S = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\,(A\wedge dA)$, was studied by Schwarz [29]. Three dimensional gauge theories with the CS term added to the Yang-Mills action were introduced in [30–33].

Non-perturbative Methods and Perturbative Methods

The CS gauge theory was first analysed from a non-perturbative point of view. Witten [11] used fundamental properties of QFT, in particular the path integral formulation, which led him to establish the equivalence between vacuum expectation values (vevs) of Wilson loops and polynomial invariants like the Jones polynomial and its generalisations. An equivalent result was then obtained by Reshetikhin and Turaev [34] using quantum groups. Then there are perturbative approaches using propagators and Feynman diagram expansion familiar in perturbative QFT. The first work was done by Guadagnini *et al.* [35] in the case of $\mathcal{M} = \mathbf{S}^3$ and link $\mathcal{L}\neq\emptyset$. It was then elaborated by Bar-Natan [36]. The case $\mathcal{M}\neq\mathbf{S}^3$, $\mathcal{L}=\emptyset$ was treated by Axelrod and Singer [37, 38]. It was pointed out that the coefficients of the perturbative series correspond to Vassiliev invariants [39, 40]. A nice review of the development of perturbative methods can be found in Labastida (1999) [41].

We will present Witten's non-perturbative method and the relation of this 3-dimensional theory to the 2-dimensional Wess-Zumino-Witten (WZW) model[6] in

[6] WZW model is also known as Wess–Zumino–Novikov–Witten (WZNW) model in recognition of the work by Sergei Novikov.

Section 5. In order to do so, we need a suitable observable for the CS theory. Before discussing the observables, let us briefly digress into the topics of CFT and WZW model.

4.2 A Brief Digression: CFT and WZW Model

A **2-dimensional** conformal field theory (CFT) can be separated into a *left-moving* (holomorphic) sector and a *right-moving* (anti-holomorphic) sector. It has a **Virasoro algebra** generated by its energy-momentum tensors $T(z), \overline{T}(\overline{z})$, and a **Kac-Moody algebra** generated by its chiral currents $J^a(z), \overline{J}^a(\overline{z})$. The left-moving generators have the mode expansions $T(z) = \sum_{n\in\mathbb{Z}} L_n/z^{n+2}$ and $J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n/z^{n+1}$; similar expressions for the right-moving counterparts. Operators (or states due to the state-operator correspondence) are labelled independently by representations of the left- and right-moving Virasoro and Kac-Moody algebras. In particular, the left-moving primary fields $\phi_l(z,\overline{z})$, $l \geq 0$ in the fundamental representation $\mathcal{R}$ of $SU(N)$ are defined by L_n and J^a_n acting on it as, $L_n\phi_l = 0, L_0\phi_l = h_l\phi_l$ and $J^a_n\phi_l = 0, J^a_0\phi_l = t^a_l\phi_l$, where the eigenvalues h_l and t^a_l are respectively the **conformal weight (dimension)** of the the field ϕ_l, and the generators of $SU(N)$ for ϕ_l. Since $L_0 + \overline{L}_0$ generates dilatations, the **scaling dimension** of the field ϕ_l is $\Delta = h_l + \overline{h}_l$ ($\overline{h}_l$ is a real number, not conjugate), the eigenvalue of the dilatation operator $D = L_0 + \overline{L}_0$.

In CFT, an N-point function of primary fields is determined by a scaling function of the fields, with power laws determined by the conformal dimensions of the fields. The Operator Product Expansion (OPE) of two primary fields in the form of:

$$\phi_i(z_1,\overline{z}_1)\phi_j(z_2,\overline{z}_2) = \sum_k c_{ij}{}^k(z_{13}, z_{23}; \overline{z}_{13,}\overline{z}_{23})\phi_k(z_3,\overline{z}_3), \quad c_{ij}{}^k \neq 0, \tag{4.6}$$

suggests that the N-point function can be written as a sum of products of three-point functions, known as **conformal blocks**[7], which appear in the work of Belavin, Polyakov, and Zamolodchikov [42]. Segal [43] described these in terms of "modular functors" that canonically associate a Hilbert space to a Riemann surface. The choice of which pair of fields is being fused is arbitrary and the OPE encodes a commutative algebra called the **fusion algebra**. This requirement leads to a set of **consistency conditions** for the OPE which, in turn, implies constraints on the correlators of the CFT, as in Fig. 14. These consistency conditions, known as crossing and unitarity symmetries, impose severe constraints on the explicit forms of the conformal blocks which, in specific CFTs, are stringent enough to determine these functions completely.

A 2-dimensional CFT is characterised by the data: the central charge c, the conformal weights h_l of the primary fields ϕ_l, and the coefficients of the OPE's of the primary fields, i.e. their fusion rules. The so-called **fusion rules** determine which

[7]Conformal blocks are also called the *pants* in texts of CFT.

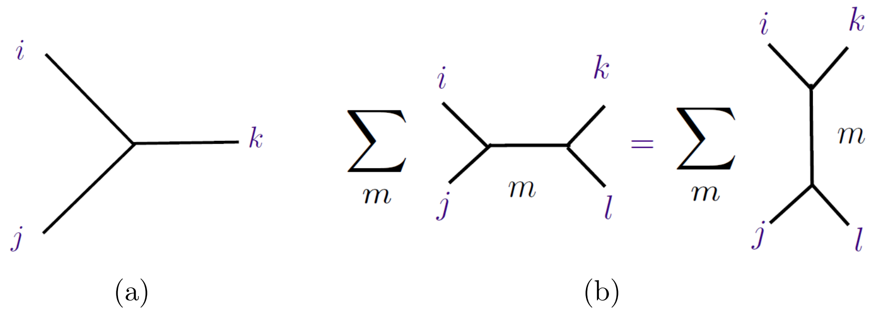


Figure 14: The OPE: (a) fusion of two primary fields, (b) consistency condition for the fusion algebra.

conformal families (primary fields and their descendants) appear in the OPE of two conformal fields. These fusion rules give constraints on the central charge and the conformal weights of the CFT. As a consequence of these constraints, it turns out that some CFT's consist of only a finite number of conformal fields. These theories are called **rational conformal field theories (RCFT's)**.

An important example of a *completely solvable, chiral* RCFT in two dimensions is the **Wess-Zumino-Witten (WZW) model** [44]. This model is a nonlinear sigma model whose action is a functional of a field Φ that maps a Riemann surface Σ to a Lie group G ($\Phi : \Sigma \to G$). It features an extended chiral algebra, called the **affine Lie algebra**, due to the existence of additional symmetries in the model. Moreover, due to the chirality property, the conformal weights of the WZW primary fields are $h_l = \overline{h}_l = \Delta$. For an extensive review of the WZW model, see, for instance, Di Francesco *et al.* (1997) [45].

4.3 Wilson Loop Operator

As a standard procedure in QFT, in addition to a Lagrangian, we wish to pick a suitable class of gauge invariant observables. In the present context, the usual gauge invariant local operators would not be appropriate, as they spoil general covariance. However, the "Wilson line" so familiar in Quantum Chromodynamics (QCD) gives a natural class of gauge invariant observables without requiring a choice of metric. Formally, the observables should satisfy

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} \rangle = 0 \tag{4.7}$$

for a set of operators $\mathcal{O}_{i_1}, \dots, \mathcal{O}_{i_n}$.

We now define the Wilson loop operator in CS theory. Quite generally, a Wilson loop operator $W_{\mathcal{R}}(C)$ in any gauge theory on a manifold $\mathcal{M}$ is described by the data of an oriented, closed curve C which is smoothly embedded in $\mathcal{M}$ and which is decorated by an irreducible representation $\mathcal{R}$ of the gauge group G. Intrinsically C is simply a circle, but the topological classification of embeddings of a circle in

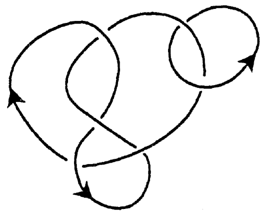


Figure 15: A link embedded in a 3-manifold.

$\mathcal{M}$ can be very complicated, such as in Fig. 15. This way of embedding defines the notion of knots and links, denoted by $\mathcal{K}$ and $\mathcal{L}$, respectively (see Section 3 for an exposition of the knot theory). From now on, instead of circles C, we will consider knots and links (in fact, circle is a type of knot, known as the unknot).

Following the standard practice in QFT, we describe the holonomy around the knot:

$$U_{\mathcal{K}} = \mathrm{Hol}_{\mathcal{K}}(A[\mathcal{K}(x)]) \tag{4.8}$$

in terms of a path-ordered exponential $\mathrm{P}\exp\Big(\oint_{\mathcal{K}} A[\mathcal{K}(x)]\Big)$, $x \in [0,1]$, which describes solutions to the first-order differential equation for for a vector $\nu(x)$, parallelly transported along $\mathcal{K}(x)$:

$$D_{\mathcal{K}}\nu(x) = \frac{d\nu(x)}{dx} + (A[\mathcal{K}(x)])\nu(x) = 0. \tag{4.9}$$

To see this, one can recursively replacing the solution

$$\nu(x+\delta x) = (1 - \delta x\, A[\mathcal{K}(x)])\nu(x) \approx \exp(-\delta x\, A[\mathcal{K}(x)])\nu(x) \tag{4.10}$$

into itself, and take $\delta x \to 0^+$, $N\delta x = x$ when reaching $\nu(x + N\delta x)$ at the end of computation. One will obtain an explicit expression for how the vector $\nu(x)$ acts under parallel transport:

$$\nu(x) = \mathrm{P}\exp(-\int_0^x dx'\, A[\mathcal{K}(x')])\, \nu(0). \tag{4.11}$$

Here, P is the path-ordering operator[8] which is essentially defined as the time-ordering operator in QFT. As a functional of the connection A, the **Wilson loop operator** $W^{\mathcal{K}}_{\mathcal{R}}(A)$ is then given by the trace in $\mathcal{R}$ of the holonomy of A around an oriented knot $\mathcal{K}$,

$$\begin{aligned} W^{\mathcal{K}}_{\mathcal{R}}(A) &= \mathrm{Tr}_{\mathcal{R}}\, U_{\mathcal{K}} = \mathrm{Tr}_{\mathcal{R}}\, \mathrm{Hol}_{\mathcal{K}}(A) \\ &= \mathrm{Tr}_{\mathcal{R}}\, \mathrm{P}\exp(\oint_{\mathcal{K}} A) \\ &= \mathrm{Tr}_{\mathcal{R}}\, \mathrm{P}\exp(\oint_{\mathcal{K}} A_i\, dx^i). \end{aligned} \tag{4.12}$$

[8]The path-ordering operator P acts on the power series expansion of the exponential by ordering the terms with higher values of t to the left.

Due to cyclicity of the trace, it is gauge invariant[9]. Physically, it characterises a charged particle moving round a loop and plays an important role in understanding the topology of the manifold.

The knots (and links) that we considered are oriented. So, if $\mathcal{K}^{-1}$ denotes the knot obtained from $\mathcal{K}$ by inverting its orientation, we have that

$$\mathrm{Tr}_{\mathcal{R}}\, U_{\mathcal{K}^{-1}} = \mathrm{Tr}_{\mathcal{R}}\, U_{\mathcal{K}}^{-1} = \mathrm{Tr}_{\overline{\mathcal{R}}}\, U_{\mathcal{K}}, \tag{4.13}$$

where $\overline{\mathcal{R}}$ denotes the conjugate representation of the group G.

With the Wilson loop operator in hand, we can compute the **partition function** (or unnormalised Wilson loop path integral) of link $\mathcal{L}$ in $\mathcal{M}$, denoted by

$$Z_{\mathcal{R}_1 \dots \mathcal{R}_L}(\mathcal{L}) = \int [\mathcal{D}A]\, e^{iS}\, (\prod_{\alpha=1}^{L} W_{\mathcal{R}_\alpha}^{\mathcal{K}_\alpha}) \tag{4.14}$$

for a link $\mathcal{L}$ with components $\mathcal{K}_\alpha$, $\alpha = 1, \dots, L$. Here, $[\mathcal{D}A]$ is a fictitious diffeomorphism invariant measure on all A's modulo gauge transformation. For a manifold $\mathcal{M}$ that contains no links, the partition function is then $Z(\mathcal{M}) = \int [\mathcal{D}A]\, e^{iS}$. We can further compute the **correlation function** (or absolutely normalised Wilson loop path integral), defined by

$$W_{\mathcal{R}_1 \dots \mathcal{R}_L}(\mathcal{L}) =< W_{\mathcal{R}_1}^{\mathcal{K}_1} \dots W_{\mathcal{R}_L}^{\mathcal{K}_L} >= \frac{1}{Z(\mathcal{M})} \int [\mathcal{D}A]\, e^{iS}\, (\prod_{\alpha=1}^{L} W_{\mathcal{R}_\alpha}^{\mathcal{K}_\alpha}). \tag{4.15}$$

Classically, the Wilson loop operator is naturally a topological invariant of the link $\mathcal{L}$ as it is defined without using any metric on the 3-manifold. However, the quantum theory may not preserve this invariance, since there would be anomalies that spoil the classical symmetry in the correlation function (4.15). Witten showed that the topological invariance of CS theory can be preserved at the quantum level, but with an extra subtlety: the invariant depends not only on the 3-manifold but also on a choice of *framing* (simply means the trivialisation of the tangent bundle $T\mathcal{M} \oplus T\mathcal{M}$).

4.4 Properties of Chern-Simons Theory

To end this section, we mention several important properties of the CS theory:

1. The CS action is purely topological, as it depends only on the choice of an orientation, not a metric on $\mathcal{M}$.

2. The classical equations of motion of the CS theory are satisfied if and only if the curvature F vanishes everywhere on $\mathcal{M}$, in which case the connection A is

[9] Generally, we have a Wilson line which is defined on an arbitrary path and is not gauge invariant. We may drop the negative sign in the Wilson loop by picking an orientation.

said to be flat. Recall that the equations of motion are the critical or stationary points of the CS action. To see this, we compute the variation of the CS action

$$\begin{aligned}\delta S &= \delta\bigg(\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\bigg)\\ &= 2\Big(\frac{k}{4\pi}\Big)\int_{\mathcal{M}} \mathrm{Tr}\,\Big((dA + A\wedge A)\wedge\delta A\Big),\end{aligned} \tag{4.16}$$

it only vanishes if $F\equiv dA+A\wedge A=0$ for all variations δA. Hence the classical solutions to CS theory are the flat connections of principal G-bundles on $\mathcal{M}$.

Flat connections are determined entirely by holonomies around non-contractible cycles on $\mathcal{M}$. In fact, the gauge equivalence classes of flat connections are in one-to-one correspondence with equivalence classes of homomorphisms from $\pi_1(\mathcal{M})$ to G up to conjugation. The classical solutions of CS over $\mathcal{M}$ is therefore the moduli space $\mathrm{Hom}(\pi_1(\mathcal{M}),G)/G$, where G acts by conjugation.

3. The CS term[10] in the action preserves local (homotopically trivial) gauge invariance for spacetime manifolds without boundary, but is not invariant under "large" (homotopically nontrivial) gauge transformations. To see this, we let $\mathcal{A}$ be the space of connections A on the G-bundle and the gauge group $\mathcal{G}=\{\mathcal{M}\to G\}$ acts on $\mathcal{A}$ via

$$A_i \to gA_ig^{-1} - \partial_i g g^{-1}, \quad g\in\mathcal{G}. \tag{4.17}$$

The CS action transforms as

$$S\to S+\frac{k}{4\pi}\int d^3x\left[\epsilon^{ijk}\partial_i\,\mathrm{Tr}(\partial_j g g^{-1}A_k)+\frac{1}{3}\epsilon^{ijk}\,\mathrm{Tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g)\right]. \tag{4.18}$$

The second term is a total derivative that vanishes but the last term does not vanish. This means that the CS action is not invariant under these large gauge transformations where the gauge transformations "wind" around spacetime. The winding is counted by the function

$$w(g)=\frac{1}{24\pi^2}\int d^3x\;\epsilon^{ijk}\;\mathrm{Tr}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g), \tag{4.19}$$

called the **winding number** $w(g)$, which appears in the last term up to a constant. Using the fact that the homotopy group $\pi_3(G)\simeq\mathbb{Z}$ for any compact connected simple Lie group G [46], the value of $w(g)$ is an integer which coincides with the index of the homotopy class of $\mathcal{G}$. We then see that the CS action shifts by integral multiples of 2π:

$$S[A]\to S[A]+2\pi k w(g). \tag{4.20}$$

[10] The CS term is also known as a topological mass term.

As in Dirac's famous work on magnetic monopoles [47], consistency in QFT does not require the gauge invariance of $S[A]$, but only of the path integral $\exp(iS[A])$. This holds if the CS level k is quantised to be an integer. This is the *quantisation* condition for CS theory first discussed in [32].

Notice that in the abelian case with compact group manifold $G = U(1)$ all the homotopy groups higher than π_1 are trivial and thus the CS coupling constant can take arbitrary values. Also, for non-compact Lie groups, there is no such quantisation condition. We will see later that the level k of CS theory (which is also the level of affine Lie algebra) is closely related to the central charge in the representation theory of affine Lie algebra.

4. The partition function of CS theory is shown [11] to be a topological invariant of an oriented, framed 3-manifold $\mathcal{M}$, at least for the weak coupling limit (large k limit). It is necessary to specify the framing of the 3-manifold, which means that the 3-manifold must be presented with a homotopy class of trivialisations of the tangent bundle.

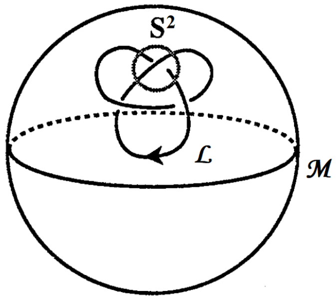


Figure 16: A link $\mathcal{L}$ on a general 3-manifold $\mathcal{M}$. A small sphere $\mathbf{S}^2$ is drawn about an inconvenient crossing.

5 Witten's Ideas and Computations

5.1 Knot and Link Invariants from Chern-Simons Theory

We will now present how the *skein relation* of the Jones polynomial emerges from CS theory with gauge group $SU(2)$. Consider an arbitrary 3-manifold $\mathcal{M}$ (specifically a 3-sphere $\mathbf{S}^3$ in later discussion) with a link $\mathcal{L}$ embedded in it, as depicted in Fig. 16.

Surgery I: Cut the Link

The Wilson loops are all in the fundamental representation (or the defining 2-dimensional representation[11]) of $SU(2)$, denoted here as $\mathcal{R}$. We identify a crossing of $\mathcal{L}$ and we surround it with a sphere $\mathbf{S}^2$, which we will cut it out and study it in detail. We can then think of the 3-manifold $\mathcal{M}$ as the connected sum: $\mathcal{M} = \mathcal{M}_L \# \mathcal{M}_R$ where $\mathcal{M}_L$ and $\mathcal{M}_R$ are the two disjoint subsets of $\mathcal{M}$ with boundary $\mathbf{S}^2$, corresponding to the outside and inside of the $\mathbf{S}^2$ respectively. By cutting the sphere, we perform a **Heegaard splitting**[12] of the 3-manifold, leading to a simple piece $\mathcal{M}_R = B$ where B is the 3-ball bounded by $\mathbf{S}^2$, and a complicated piece $\mathcal{M}_L = \mathcal{M}\backslash B$, as depicted in Fig. 17. Notice that the boundaries of $\mathcal{M}_R$ and $\mathcal{M}_L$ are both $\mathbf{S}^2$, $\partial\mathcal{M}_R = \partial\mathcal{M}_L = \mathbf{S}^2$, but they have opposite orientation. On the boundary, there are four **marked points** indicating the intersections of $\mathcal{L}$ with $\mathbf{S}^2$ which are connected by two lines in the interior of the 3-ball.

[11]The defining representation is a matrix subgroup of the Lie group G whose basis space is formed by N real or complex vectors. This is generally different from the fundamental representation, which is a finite dimensional irreducible representation of a semisimple Lie group or Lie algebra if its highest weight is a fundamental weight. For classical algebras $\mathfrak{su}(N), \mathfrak{so}(N), \mathfrak{sp}(N)$ and their groups, the defining representation is a fundamental representation, which has been used interchangeably in most of the physics literature.

[12]A Heegaard splitting is a decomposition of a compact oriented 3-manifold into two handlebodies. Their common boundary is called the Heegaard surface of the splitting. Splittings are considered up to ambient isotopy.

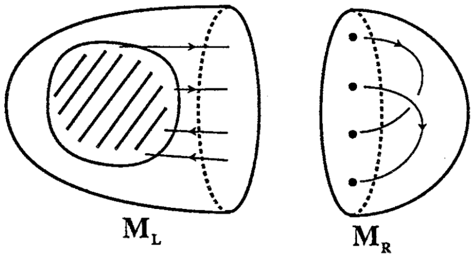


Figure 17: Manifold $\mathcal{M}$ is split into $\mathcal{M}_R$ (the interior of $\mathbf{S}^2$) and $\mathcal{M}_L$ (the exterior of $\mathbf{S}^2$ with complicated piece whose details are not drawn).

Canonical Quantisation

To each boundary $\partial\mathcal{M}_R$ and $\partial\mathcal{M}_L$, we can associate with them the 2-dimensional physical Hilbert spaces $\mathcal{H}_R$ and $\mathcal{H}_L$ (canonically dual to one another). This association is called the **canonical quantisation** of path integrals of CS theory over a manifold (boundary) [8]. Then, the path integrals on $\mathcal{M}_R$ and $\mathcal{M}_L$ give vectors ψ and χ in $\mathcal{H}_R$ and $\mathcal{H}_L$, respectively.

If we act on the boundary $\partial\mathcal{M}_R$ with a diffeomorphism K before gluing the manifolds back, then the vector ψ in $\mathcal{H}_R$ is replaced by $K\psi$ so (χ, ψ) is replaced by $(\chi, K\psi)$. This potentially provides a way to determine how the partition function of a TQFT transforms under *surgery*.

Surgery II: Glue the Link

By gluing back the manifolds $\mathcal{M}_L$ and $\mathcal{M}_R$ back together, we will have the partition function of link $\mathcal{L}$ that we want, written as

$$Z(\mathcal{M};\mathcal{L}) = (\chi, \psi), \tag{5.1}$$

where the RHS is a natural pairing of the vectors, meaning that for any nonzero vector $\psi \in \mathcal{H}_R$ there is a vector $\chi \in \mathcal{H}_L$ such that $(\chi, \psi) \neq 0$, and vice-versa [48]. There is no way to evaluate (5.1) since we know neither ψ nor χ. The one thing that we know at present is the pairing occurs in a 2-dimensional vector space. We can use the marvelous property that in a 2-dimensional vector space, any three vectors ψ, ψ_1, ψ_2 in $\mathcal{H}_R$ obey a relation of linear dependence, i.e.

$$\alpha\psi + \beta\psi_1 + \gamma\psi_2 = 0, \tag{5.2}$$

where α, β, and γ are complex functions. We can achieve this relation by considering two diffeomorphisms[13] of $\mathcal{M}_R$, namely X_1 and X_2 as in Fig. 18, that swap the marked

[13] Let us call the modified manifold, in which a diffeomorphism is applied, $\mathcal{M}'_R$. When we put $\mathcal{M} = \mathcal{M}_L \# \mathcal{M}'_R$ back together, we would have changed the equivalence class of the knot, because the "braids" connecting the marked points would have modified the crossing. The expansion of a link in terms of two variations of that same link is exactly the skein relation (3.12).

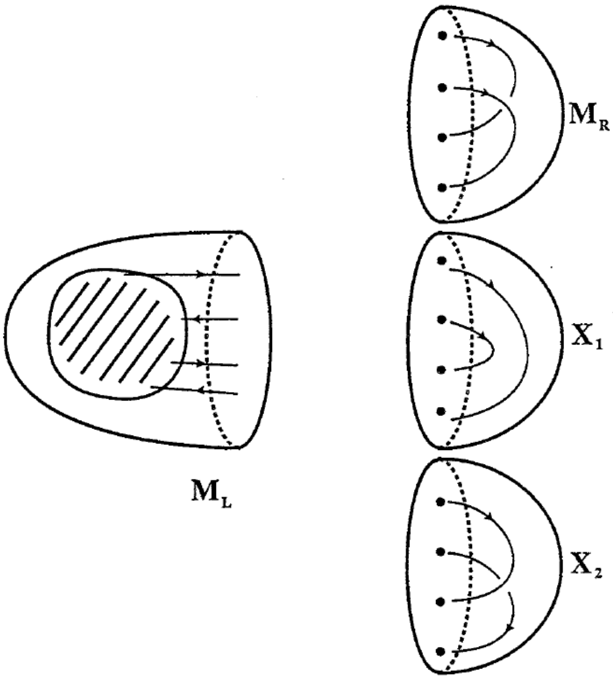


Figure 18: Manifold $\mathcal{M}_R$ and its variations.

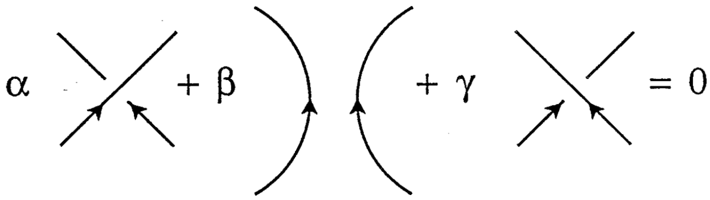


Figure 19: A link expectation value relation.

points in such a way that they result in the other two crossings from the *oriented Conway triple* (Fig. 11). Let ψ_1 and ψ_2 be the corresponding vectors of X_1 and X_2 in $\mathcal{H}_R$, we can then write a linear dependence as in (5.2). It follows immediately that,

$$\alpha(\chi, \psi) + \beta(\chi, \psi_1) + \gamma(\chi, \psi_2) = 0, \tag{5.3}$$

and hence,

$$\alpha Z(\mathcal{L}) + \beta Z(\mathcal{L}_1) + \gamma Z(\mathcal{L}_2) = 0, \tag{5.4}$$

where the notations of manifolds are omitted to make the resemblance to the skein relation (3.12) of Jones polynomial clearer. This relation (5.4) is often drawn as in Fig. 19. In fact, it uniquely determines the *expectation values* of all knots and links in $\mathbf{S}^3$ (see Section 4.1 in [11] for proof).

Half-Monodromy

What is left now is to compute the coefficients α, β, and γ. This requires a further study of the 2-dimensional Hilbert space $\mathcal{H}_R$ which arises as a 1-dimensional space of **conformal blocks** for the $\mathcal{R}, \mathcal{R}, \overline{\mathcal{R}}, \overline{\mathcal{R}}$ four-point functions on $\mathbf{S}^2$ ($\overline{\mathcal{R}}$ being the dual of the defining 2-dimensional representation of $SU(2)$, $\mathcal{R}$). The three configurations in Fig. 18 can be regarded as differing from each other by a diffeomorphism called the **half-monodromy** acting on the boundary $\mathcal{M}_R$ under which the two copies of $\mathcal{R}$ switch places by taking a half-step around one another. This is indicated in Fig. 20. These half-monodromies may be represented as a linear transformation on the state (vector) ψ such that,

$$\psi_1 = B\psi, \quad \psi_2 = B^2\psi. \tag{5.5}$$

Since the matrix B acts in a 2-dimensional space, it obeys a characteristic equation

$$B^2 - \text{tr}(B)B + \det(B) = 0, \tag{5.6}$$

which after multiplying everywhere by ψ, we obtain

$$B^2\psi - \text{tr}(B)B\psi + \det(B)\psi = 0, \tag{5.7}$$

or equivalently,

$$\psi_2 - \text{tr}(B)\psi_1 + \det(B)\psi = 0. \tag{5.8}$$

Making a natural pairing for each term and rearranging the equation into the form of (5.4), we have

$$\alpha = \det(B), \quad \beta = -\,\text{tr}(B), \quad \gamma = 1. \tag{5.9}$$

Now we need only to know the eigenvalues (and hence the determinant and trace) of B.

Eigenvalues of Half-monodromy

We will quote the results for the eigenvalues of B from Moore and Seiberg [49]. Before discussing the formulae, one important thing to take into account is that all concrete results including the values of α, β, and γ depend on the framing of links. The convention in Moore and Seiberg's results is that one should pick the same framing for each diagram on the right in Fig. 18, for instance, a *canonical framing* in which a unit vector coming out of the page defines a normal vector field on each link in the diagrams.

Consider the Lie group $G = SU(N)$, the eigenvalues of B are

$$\lambda_i = \pm \exp(i\pi(2h_{\mathcal{R}} - h_{E_i})). \tag{5.10}$$

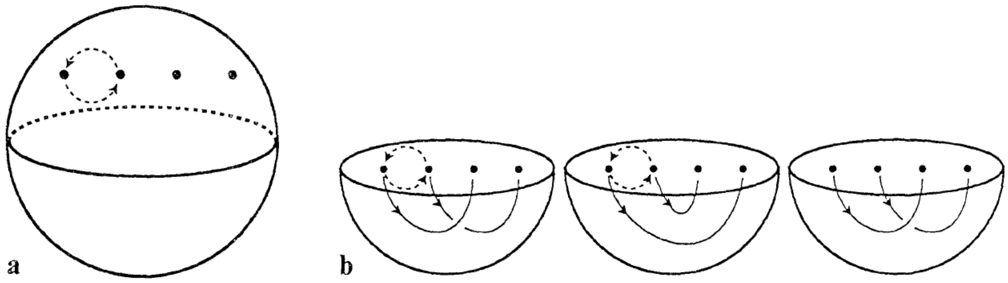


Figure 20: (a) The half-monodromy operation exchanges two equivalent points on $\mathbf{S}^2$, the arrows indicate the process in which the first two points switch places by executing a half-twist about one another. (b) When the two points on the left undergo a half-twist about one another, the first diagram becomes the second, and when this is done again, the second diagram becomes the third. These are clearly the three diagrams $\mathcal{M}_R$, X_1, and X_2 of Fig. 18 which differ by a succession of half-monodromies.

Here $h_{\mathcal{R}}$ is the **conformal weight** of the WZW primary field corresponding to the representation $\mathcal{R}$:

$$h_{\mathcal{R}} = \frac{N^2-1}{2N(k+N)}; \tag{5.11}$$

h_{E_i} is are the weights of the corresponding primary fields where E_i are the irreducible representations of $SU(N)$ appearing in the decomposition of $\mathcal{R}\otimes\mathcal{R}$: $\mathcal{R}\otimes\mathcal{R} = \bigoplus_{i=1}^{s} E_i$. The physical Hilbert space $\mathcal{H}$ at large k level is s-dimensional. In this case, $s=2$, then $\mathcal{H}$ is 2-dimensional (except for $k=1$ where it is 1-dimensional). The $+$ or $-$ sign corresponds to whether E_i appears symmetrically or antisymmetrically in $\mathcal{R}\otimes\mathcal{R}$. If $\mathcal{R}$ is the N-dimensional representation of $SU(N)$, then in the decomposition $\mathcal{R}\otimes\mathcal{R}$, the symmetric piece is an irreducible representation with $h_{E_1} = \frac{N^2+N-2}{N(k+N)}$, and the antisymmetric piece is an irreducible representation with $h_{E_2} = \frac{N^2-N-2}{N(k+N)}$. One will find that the eigenvalues of B are

$$\lambda_1 = \exp\left(\frac{i\pi(-N+1)}{N(k+N)}\right), \quad \lambda_2 = -\exp\left(\frac{i\pi(N+1)}{N(k+N)}\right). \tag{5.12}$$

By plugging these eigenvalues into (5.9), one finds that

$$\alpha = -\exp\left(\frac{2i\pi}{N(k+N)}\right), \quad \beta = -\exp\left(\frac{i\pi(-N+1)}{N(k+N)}\right) + \exp\left(\frac{i\pi(N+1)}{N(k+N)}\right), \quad \gamma = 1. \tag{5.13}$$

Dehn Diffeomorphism

The half-monodromy B induces a twist in the framed link segments. This is illustrated in Fig. 21. We see that the framing is not the standard framing that we

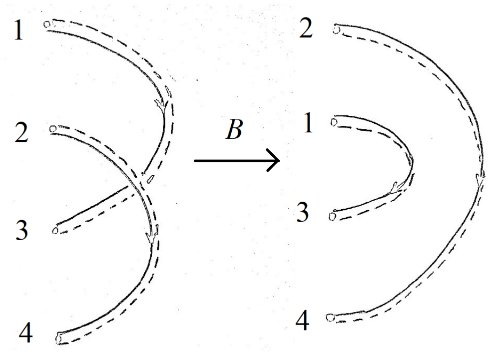


Figure 21: A twist is induced in the framing after the operation of half-monodromy B.

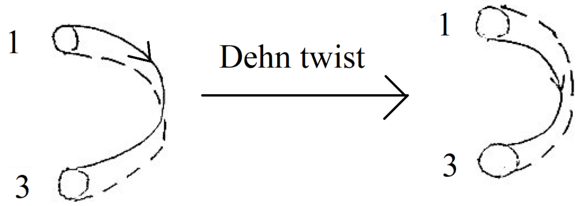


Figure 22: A Dehn twist reverts the framed link to its original framing.

had. Before "gluing" back the manifolds in Fig. 18, we need to fix this by applying another kind of diffeomorphism, called a **Dehn twist**, D, to reverse this unwanted twisting, as indicated in Fig. 22. In conformal field theory, the t-fold Dehn twist acts on the Hilbert space $\mathcal{H}_R$ as a multiplication by $\exp(-2\pi i t h_{\mathcal{R}})$, where $h_{\mathcal{R}}$ is again the conformal weight of the WZW primary field in the $\mathcal{R}$ representation. The negative sign indicates the reverse action.

We know that $\psi_1 = B\psi$ in which ψ has a twist in its framing. To untwist ψ_1 is equivalent to multiplying the coefficient of ψ_1, β by $\exp(-2\pi i h_{\mathcal{R}})$. Similarly, due to $\psi_2 = B^2\psi$, ψ_2 has to be untwisted twice and we have to multiply γ by $\exp(-4\pi i h_{\mathcal{R}})$. After these corrections, we get

$$
\begin{aligned}
\alpha &= -\exp\left(\frac{2\pi i}{N(k+N)}\right), \\
\beta &= -\exp\left(\frac{\pi i(2-N-N^2)}{N(k+N)}\right) + \exp\left(\frac{\pi i(2+N-N^2)}{N(k+N)}\right), \\
\gamma &= \exp\left(\frac{2\pi i(1-N^2)}{N(k+N)}\right).
\end{aligned}
\tag{5.14}
$$

We can simplify all coefficients by multiplying them with a common factor $\exp(\frac{i\pi(N^2-2)}{N(k+N)})$ and making the substitution $t = \exp(\frac{2\pi i}{k+N})$. We will end up with the Jones polynomial skein relation,

$$-t^{N/2}Z(\mathcal{L}_+) + (t^{1/2} - t^{-1/2})Z(\mathcal{L}_0) + t^{-N/2}Z(\mathcal{L}_-) = 0, \tag{5.15}$$

where we have used the standard notations of overcrossing, zero crossing, and undercrossing: $\mathcal{L}_+$, $\mathcal{L}_0$, and $\mathcal{L}_-$ to replace $\mathcal{L}$, $\mathcal{L}_1$, and $\mathcal{L}_2$, respectively.

When we restrict to the $SU(2)$ case, the skein relation is

$$-tZ(\mathcal{L}_+) + (t^{1/2} - t^{-1/2})Z(\mathcal{L}_0) + t^{-1}Z(\mathcal{L}_-) = 0, \tag{5.16}$$

which agrees with (3.12), with the identification,

$$V(\mathcal{L}_i) = \frac{Z(\mathcal{L}_i)}{Z(\text{unknotted Wilson loop})} \quad i = +, 0, -. \tag{5.17}$$

The precise relation between the Jones polynomial and the correlation function of Wilson loops, for any links $\mathcal{L}$ in $\mathbf{S}^3$, in $SU(2)$ fundamental representation $\mathcal{R} = \square$ (in terms of Young tableau), is given by

$$W_{\square\ldots\square}(\mathcal{L}) = t^{2\,lk(\mathcal{L})}\left(\frac{t - t^{-1}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right)V^{\mathcal{L}}(t), \tag{5.18}$$

as a special case of (3.18). In a similar fashion, the partition function of Wilson loops $Z(\mathcal{L})$ in CS theory, with different gauge groups, can be used to compute other knot and link polynomial invariants.

Correlation Function of Unknot

We would like to compute the correlation function of an unknotted Wilson loop C, $W_\square(C)$, in $\mathbf{S}^3$. Recall that an unknot has no crossings. Although this may seem a trivial case, it contains some essential features.

The equation (5.4) (and Fig. 19) reduces to the case with a certain number of unlinked and unknotted circles. This can be sketched as in Fig. 23. The first and third links consist of a single unknotted circle (unknotted by Reidemeister moves), and the second consists of two unknotted circles that are unlinked. If we denote the partition function for s unlinked and unknotted circles C in the fundamental representation $\square$ of $SU(N)$ as $Z(\mathbf{S}^3; C^s)$, then (5.4) amounts to the assertion that

$$(\alpha + \gamma)Z(\mathbf{S}^3; C) + \beta Z(\mathbf{S}^3; C^2) = 0. \tag{5.19}$$

To get the corresponding correlation function

$$W_\square(C) = Z(\mathbf{S}^3; C)/Z(\mathbf{S}^3), \tag{5.20}$$

we divide throughout (5.19) by partition function for the 3-sphere $\mathbf{S}^3$ without links, $Z(\mathbf{S}^3)$, and obtain,

$$(\alpha + \gamma)W_\square(C) + \beta W_\square(C^2) = 0. \tag{5.21}$$

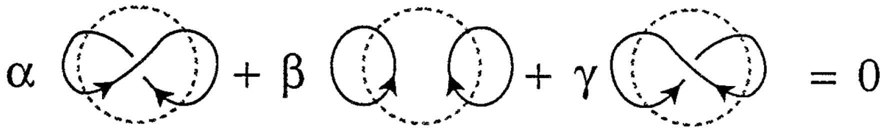


Figure 23: The three diagrams are identical outside of the dotted lines, and look like Fig. 19 inside them.

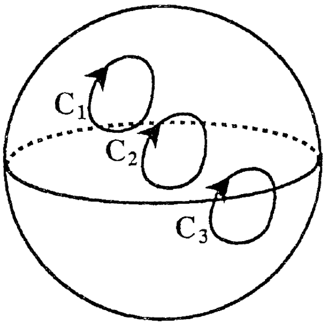


Figure 24: A 3-sphere $\mathbf{S}^3$ with three unlinked and unknotted circles C_1, C_2, C_3 associated with representations $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$.

What is the correlation function of the two disjoint circles, $W_\square(C^2)$? To answer this question, we use the following relation. For an arbitrary collection of unlinked, unknotted Wilson loops on $\mathbf{S}^3$, as illustrated in Fig. 24, we write the partition function of $\mathbf{S}^3$ with this collection of Wilson loops as $Z(\mathbf{S}^3;\, C_1 \dots C_s)$. By cutting $\mathbf{S}^3$ in various ways to separate the circles, we learn that[14]

$$\frac{Z(\mathbf{S}^3;\, C_1 \dots C_s)}{Z(\mathbf{S}^3)} = \prod_{k=1}^{s} \frac{Z(\mathbf{S}^3;\, C_k)}{Z(\mathbf{S}^3)}. \tag{5.22}$$

and similarly,

$$W_\square(C_1 \dots C_s) = \prod_{k=1}^{s} W_\square(C_k). \tag{5.23}$$

Using relation (5.23), we have $W_\square(C^2) = (W_\square(C))^2$. This implies that the correlation function of the unknotted Wilson loop in $SU(N)$ fundamental representation $\square$ is

$$W_\square(C) = -\frac{\alpha+\gamma}{\beta}, \tag{5.24}$$

[14]The result in (5.22) is deduced from a general formula $Z(\mathcal{M}) \cdot Z(\mathbf{S}^3) = Z(\mathcal{M}_1) \cdot Z(\mathcal{M}_2)$, which is also expressed as $\frac{Z(\mathcal{M})}{Z(\mathbf{S}^3)} = \frac{Z(\mathcal{M}_1)}{Z(\mathbf{S}^3)} \cdot \frac{Z(\mathcal{M}_2)}{Z(\mathbf{S}^3)}$. This is proved in the introductory part of Section 4 in Witten (1989) [11].

which is a rational function in terms of the polynomials α, β, and γ. By comparing (5.24) and (5.15), we finally have

$$W_{\Box}(C) = \frac{t^{N/2} - t^{-N/2}}{t^{1/2} - t^{-1/2}}. \tag{5.25}$$

One can easily check that the equation (5.25) for $N = 2$ can also be obtained through (5.18) with the unknot satisfying $lk(C) = 0, V^C(t) = 1$, in $SU(2)$ case.

There are several interesting facts about the formula (5.25). First, this formula is positive for all values of the CS variables N and k. This is required by reflection positivity of the CS gauge theory in three dimensions. Second, in the weak coupling limit of $k \to \infty$, we have $W_{\Box}(C) \to N$. This is easily interpreted; in the weak coupling limit, the fluctuations in the connection A_i on $\mathbf{S}^3$ are irrelevant, and the vacuum expectation value (vev) of the Wilson loop approaches its value for $A_i = 0$, which is the dimension of the representation, or in this case N.

5.2 Framing Dependence of Knots and Links

There is still one ambiguity in the results that we obtained. We have to consider the self-linking numbers (i.e. framings) of links and knots in $\mathbf{S}^3$, which will have effect on their correlation functions. For the notions of framing, readers may refer to Section 3.3.

Abelian Chern-Simons Theory

We first consider the $U(1)$ abelian CS theory [50] whereby the cubic term in (4.4) drops out and only a Gaussian theory is left. By evaluating the correlation function (4.15), contractions of the holonomies corresponding to different knots $\mathcal{K}_\alpha$, $\mathcal{K}_\beta$ produces the corresponding integral (3.6). One shall also consider the contractions of the holonomies round the same knot $\mathcal{K}$, i.e. self-contractions when $\mathcal{K}_\alpha = \mathcal{K}_\beta$ (cf. (3.7)). By means of a framing, the correlation function that one obtains is a topological invariant but the price that one has to pay is that the regularisation depends on a set of cotorsion integers $p_\alpha = \phi(\mathcal{K}) = lk(\mathcal{K}_\alpha, \mathcal{K}_{f,\alpha})$ (one for each knot). The correlation function (4.15) is then

$$\langle \prod_\alpha \exp(n_\alpha \oint_{\mathcal{K}_\alpha} A) \rangle = \exp\left[\frac{\pi i}{k}\Big(\sum_\alpha n_\alpha^2\, p_\alpha + \sum_{\alpha \neq \beta} n_\alpha n_\beta\, lk(\mathcal{K}_\alpha, \mathcal{K}_\beta)\Big)\right]. \tag{5.26}$$

This regularisation is simply the "point-splitting" method in the context of QFT.

$SU(N)$ Chern-Simons Theory

We can now generalise the abelian case to nonabelian CS theory with gauge group $SU(N)$. The same kind of ambiguities arise in the self-contractions of the Wilson loop holonomies, i.e. a choice of framing has to be made for each knot $\mathcal{K}_\alpha$. The only

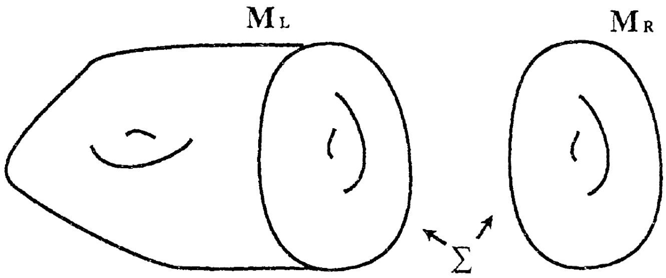


Figure 25: A 3-manifold $\mathcal{M}$ is separated into two pieces, $\mathcal{M}_L$ and $\mathcal{M}_R$ with Σ as their common boundaries.

difference from the abelian case is that the self-contraction of $\mathcal{K}_\alpha$ gives a group factor $\mathrm{Tr}_{\mathcal{R}_\alpha}(T_a T_a)$, where T_a is a basis (generator) of the Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$.

Our physical result, i.e. the evaluation of vev of the product of Wilson loops in CS theory, depends on the framing that we pick for knots and links. In general, there is no canonical framing (zero self-linking number) in the set of possible framings of knots and links in a 3-manifold. However, we are fortunate to have a canonical framing for knots and links in $\mathbf{S}^3$. If one compares two framings, they always differ by a definite integer, which is the relative twist in going around the knots or links (Fig. 8b). Hence, for any other framing of $\mathcal{K}_\alpha$ differ from the canonical framing by t_α units, the vev will pick up a phase, i.e.

$$W_{\mathcal{R}_1 \ldots \mathcal{R}_L} \to \exp\left[2\pi i \sum_{\alpha=1}^{L} t_\alpha h_{\mathcal{R}_\alpha}\right] W_{\mathcal{R}_1 \ldots \mathcal{R}_L}. \tag{5.27}$$

5.3 Surgery Relation

Let us describe the operation of surgery on links on an arbitrary 3-manifold $\mathcal{M}$ (instead of $\mathbf{S}^3$ that we have discussed). The idea here is that the computations of invariants on any 3-manifold can be reduced to $\mathbf{S}^3$. We begin with an arbitrarily selected link $\mathcal{L}$ (without any Wilson loop associated with it) embedded in a 3-manifold $\mathcal{M}$. We first thicken $\mathcal{L}$ to a *tubular neighbourhood*, a solid torus $\mathbf{U}$ centered on $\mathcal{L}$. Removing $\mathbf{U}$, $\mathcal{M}$ is Heegaard split into two pieces, with $\mathcal{M}_R$ being the solid torus $\mathbf{U}$ and $\mathcal{M}_L$ being the remainder, as indicated in Fig. 25. We then make a diffeomorphism K on the boundary of $\mathcal{M}_R$, Σ, and glues $\mathcal{M}_L$ and $\mathcal{M}_R$ back together to obtain a new 3-manifold $\tilde{\mathcal{M}}$. The aim of this section is to prove that computations on $\tilde{\mathcal{M}}$ are equivalent to computations on $\mathcal{M}$ with a physical Wilson line where the surgery was made.

Verlinde [8] showed that if Σ is a Riemann surface of genus one, then the dimension of the physical Hilbert space $\mathcal{H}_\Sigma$ is t, the number of integrable highest weight

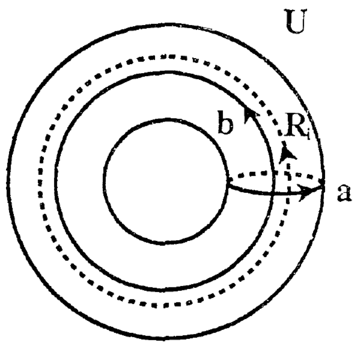


Figure 26: A basis of the physical Hilbert space is indicated consisting of states obtained by placing a Wilson loop in the $\mathcal{R}_i$ representation, in the interior of $U(=\mathcal{M}_R)$, parallel to the non-contractible cycle b, and performing the path integral to get a vector v_i in $\mathcal{H}_R$.

representations of a loop algebra[15] $L\mathfrak{g}$ at level k. Moreover, though there is no canonical basis for $\mathcal{H}_\Sigma$, Verlinde showed that every choice of a homology basis for $H^1(\Sigma;\mathbb{Z})$, consisting of two cycles a and b, gives a canonical choice of basis in $\mathcal{H}_\Sigma$. In general, the properties of $\mathcal{H}_\Sigma$ are just like those of $H^1(\Sigma;\mathbb{Z})$.

To proceed, we place a Wilson loop in $\mathcal{R}_i$ $(i=0,\ldots,t-1)$ representation in the interior of the solid torus $\mathbf{U}$, running in the direction of non-contractible cycle b, and perform the path integral in $\mathbf{U}$ to define a vector v_i in $\mathcal{H}_R$ (as indicated in Fig. 26). The v_i make up the Verlinde basis in $\mathcal{H}_R$. It is known that a Wilson loop in the trivial representation $\mathcal{R}_0$ is equal to 1, so the vector v_0 here is the same as the vector obtained by a path integral on $\mathbf{U}$ without Wilson loops. The partition function of $\mathcal{M}$ is then $Z(\mathcal{M})=(\chi,v_0)$ for a vector χ in $\mathcal{H}_L$.

A diffeomorphism K on Σ of $\mathcal{M}_R$ is represented in the Verlinde basis by an explicit matrix K_{ij}:

$$K\cdot v_i=\sum_j K_{ij}v_j \tag{5.28}$$

Gluing back to make $\tilde{\mathcal{M}}$, we have the partition function of $\tilde{\mathcal{M}}$ as $Z(\tilde{\mathcal{M}})=(\chi,Kv_0)$. Using (5.28), we can write

$$Z(\tilde{\mathcal{M}})=\sum_j K_{0j}(\chi,v_j). \tag{5.29}$$

[15] A *loop algebra* $L\mathfrak{g}$ is an infinite-dimensional algebra, defined by the tensor product of the usual Lie algebra $\mathfrak{g}$ with $C^\infty(\mathbf{S}^1)$, the algebra of smooth complex functions over $\mathbf{S}^1$: $L\mathfrak{g}=\mathfrak{g}\otimes C^\infty(\mathbf{S}^1)$, with the Lie bracket given by $[g_1\otimes f_1,g_2\otimes f_2]=[g_1,g_2]\otimes f_1f_2$ for $g_1,g_2\in\mathfrak{g}$, $f_1f_2\in C^\infty(\mathbf{S}^1)$. It can be thought of in terms of a smooth map from $\mathbf{S}^1$ to $\mathfrak{g}$, i.e. a smooth parametrised loop in $\mathfrak{g}$. More technically, one considers the *affine Lie algebra* $\hat{\mathfrak{g}}$ that is obtained by adding one extra dimension to the loop algebra $L\mathfrak{g}$ and modifying the Lie bracket in a nontrivial way, which physicists call a *quantum anomaly* of the WZW model and mathematicians a *central extension*.

Just as (χ, v_0) represents the original partition function of $\mathcal{M}$, we can have (χ, v_j) representing $Z(\mathcal{M}; \mathcal{R}_j)$, a modified partition function of $\mathcal{M}$ with an extra Wilson loop in the $\mathcal{R}_j$ representation placed on $\mathcal{L}$ (in addition to whatever Wilson loops are already present on $\mathcal{M}$). Hence we can rewrite (5.29) as

$$Z(\tilde{\mathcal{M}}) = \sum_j K_{0j} Z(\mathcal{M}; \mathcal{R}_j). \tag{5.30}$$

We have so far only considered a link $\mathcal{L}$ without a Wilson loop before the surgery. What if before the surgery a Wilson loop in the $\mathcal{R}_i$ representation was already present on $\mathcal{L}$? Surgery amounts to cutting out a tubular neighbourhood $U = \mathcal{M}_R$ of $\mathcal{L}$ and then gluing it back in, and after this process the $\mathcal{R}_i$ Wilson loop will still be present in $\tilde{\mathcal{M}}$. So the left-hand side of (5.30) is replaced by $Z(\tilde{\mathcal{M}}; \mathcal{R}_i)$. We now consider the right-hand side of (5.30). Before surgery, with a Wilson loop $\mathcal{R}_i$ on $\mathcal{L}$, the path integral on $\mathcal{M}$ of $\mathcal{L}$ gives on the boundary Σ a state v_i (a generalisation of v_0). When we cut out $\mathbf{U}$ and glue it back in with a diffeomorphism K on Σ, then v_i is replaced with $K_{ij} v_j$. So the right-hand side of (5.30) becomes $\sum_j K_{ij} Z(\mathcal{M}; \mathcal{R}_j)$ and we get the **generalised surgery relation**,

$$Z(\tilde{\mathcal{M}}; \mathcal{R}_j) = \sum_j K_{ij} Z(\mathcal{M}; \mathcal{R}_j). \tag{5.31}$$

The matrix K_{ij} is expressed in terms of the irreducible level k representations of $L\mathfrak{g}$ (or the integrable representations of the affine Lie algebra $\hat{\mathfrak{g}}$).

5.4 Manifold Invariant from Chern-Simons Theory

We have so far left out the partition function of $\mathbf{S}^3$ without Wilson loops, $Z(\mathbf{S}^3)$, in our discussion. In this section, we are going to determine it. It may come as a surprise to topologists that we cannot trivially assert that this is 1.

Partition Function of $\mathbf{S}^2 \times \mathbf{S}^1$

In TQFT, there are no particularly strong axioms governing $Z(\mathbf{S}^3)$. There are, however, axioms for partition function of manifold of the form $X \times \mathbf{S}^1$, for various manifolds X. The upshot is that $\mathbf{S}^3$ can be obtained from $\mathbf{S}^2 \times \mathbf{S}^1$ through a surgery, which we will later explain.

Our starting point is to study $X \times \mathbf{S}^1$ in a Hamiltonian formalism, as indicated in Fig. 27a. We construct a physical Hilbert space $\mathcal{H}_X$ of X. Then, we introduce a "time" t direction, represented by a unit interval $[0, 1]$, which we will propagate the vectors in $\mathcal{H}_X$ from $t = 0$ to $t = 1$. This operation is trivial, since the Chern-Simons theory, like any generally covariant theory, has a vanishing Hamiltonian. Finally, we form $X \times \mathbf{S}^1$ by joining $X \times \{0\}$ to $X \times \{1\}$. This identifies the initial and final states, giving a trace:

$$Z(X \times \mathbf{S}^1) = \mathrm{Tr}_{\mathcal{H}_X}(\mathbf{1}) = \dim(\mathcal{H}_X). \tag{5.32}$$

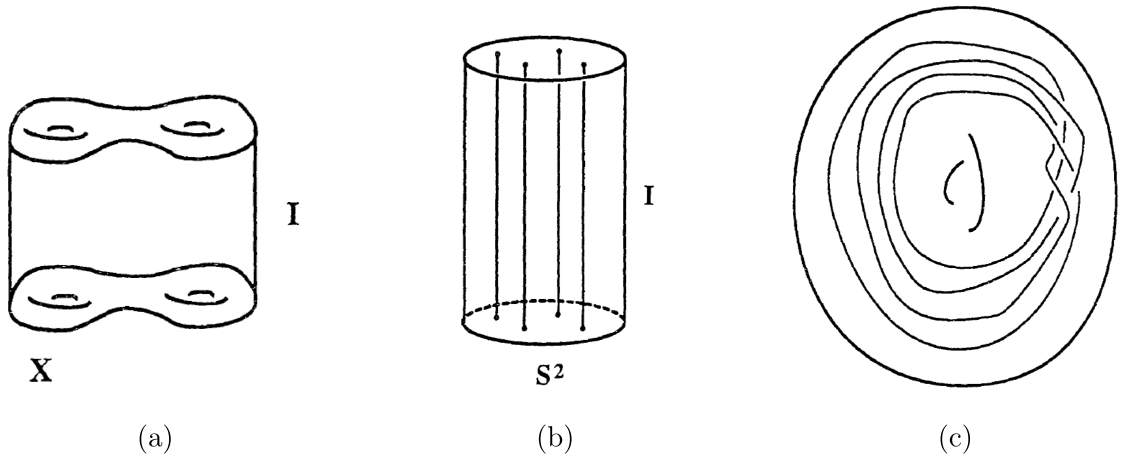


Figure 27: (a) Beginning with $X \times I$, one makes $X \times \mathbf{S}^1$ by identifying $X \times \{0\}$ with $X \times \{1\}$. Diagram in (b) is constructed when X is $\mathbf{S}^2$ with some marked points p_i. If $\mathbf{S}^2 \times \{0\}$ is glued to $\mathbf{S}^2 \times \{1\}$ via a nontrivial diffeomorphism B, one makes in this way $\mathbf{S}^2 \times \mathbf{S}^1$ with a *braid*, as in (c).

The trace of the identity operator $\mathbf{1}$ in $\mathcal{H}_X$ is due to the use of the *canonical framing* of the Wilson loop which is invariant under rotations of $\mathbf{S}^1$. For instance, the physical Hilbert space of $\mathbf{S}^2$ is 1-dimensional[16] for any Lie group G and level k, so we have

$$Z(\mathbf{S}^2 \times \mathbf{S}^1) = 1. \tag{5.33}$$

It is possible to generalise (5.32) as follows. Given a diffeomorphism $K : X \to X$, we can form the mapping cylinder $X \times_K \mathbf{S}^1$ by identifying $x \times \{1\}$ with $Kx \times \{0\}$ for every $x \in X$. In the framework of QFT, going from $X \times I$ to $X \times_K \mathbf{S}^1$, the initial and final states are identified via K, so the generalisation of (5.32) is

$$Z(X \times_K \mathbf{S}^1) = \mathrm{Tr}_{\mathcal{H}_X}(K). \tag{5.34}$$

The situation that we actually wish to consider is when X is $\mathbf{S}^2$ with some marked points p_a, $a = 1, \ldots, s$ to which representations $\mathcal{R}_{i(a)}$ are assigned (for $a = 1, \ldots, s$, $i(a)$ is one of the values $0, \ldots, t-1$ corresponding to integrable representations of the loop algebra associated to G at level k.). In particular, we have $\mathbf{S}^2 \times \mathbf{S}^1$ with some Wilson loops which are unknotted, parallel circles of the form $p_a \times \mathbf{S}^1$, as sketched in Fig. 27b and Fig. 27c. As an analogue of (5.32), the partition function of $\mathbf{S}^2 \times \mathbf{S}^1$, $Z(\mathbf{S}^2 \times \mathbf{S}^1; \langle \mathcal{R} \rangle)$, and the Hilbert space of $\mathbf{S}^2$ with charges in the representation $\mathcal{R}_{a_i}$ are related by

$$Z(\mathbf{S}^2 \times \mathbf{S}^1; \langle \mathcal{R} \rangle) = \dim(\mathcal{H}_{\mathbf{S}^2; \langle \mathcal{R} \rangle}). \tag{5.35}$$

There are some special cases which we need to consider to determine the partition function of $\mathbf{S}^3$.

[16] There are no flat connections on $\mathbf{S}^2$ and the quantisation is trivial. There is just a unique state: $\dim(\mathcal{H}_{\mathbf{S}^2}) = 1$.

1. If the collection of representations $\langle\mathcal{R}\rangle$ consists of a single representation $\mathcal{R}_a$, we get

$$Z(\mathbf{S}^2\times\mathbf{S}^1;\mathcal{R}_a)=\delta_{a,0} \tag{5.36}$$

since the physical Hilbert space with a single charge in the $\mathcal{R}_a$ representation is one dimensional if $\mathcal{R}_a$ is the trivial representation ($a=0$) and zero dimensional otherwise.

2. For two charges in the representations $\mathcal{R}_a$ and $\mathcal{R}_b$, we get

$$Z(\mathbf{S}^2\times\mathbf{S}^1;\mathcal{R}_a,\mathcal{R}_b)=g_{ab}, \tag{5.37}$$

where the "metric" g_{ab} is 1 if $\mathcal{R}_b$ is the dual of $\mathcal{R}_a$ and zero otherwise.

3. If there are three charges in the representations $\mathcal{R}_a$, $\mathcal{R}_b$, $\mathcal{R}_c$, we get

$$Z(\mathbf{S}^2\times\mathbf{S}^1;\mathcal{R}_a,\mathcal{R}_b,\mathcal{R}_c)=N_{abc} \tag{5.38}$$

with the trilinear "coupling" (or "structure constant") N_{abc} of Verlinde [8], which is the dimension of the physical Hilbert space.

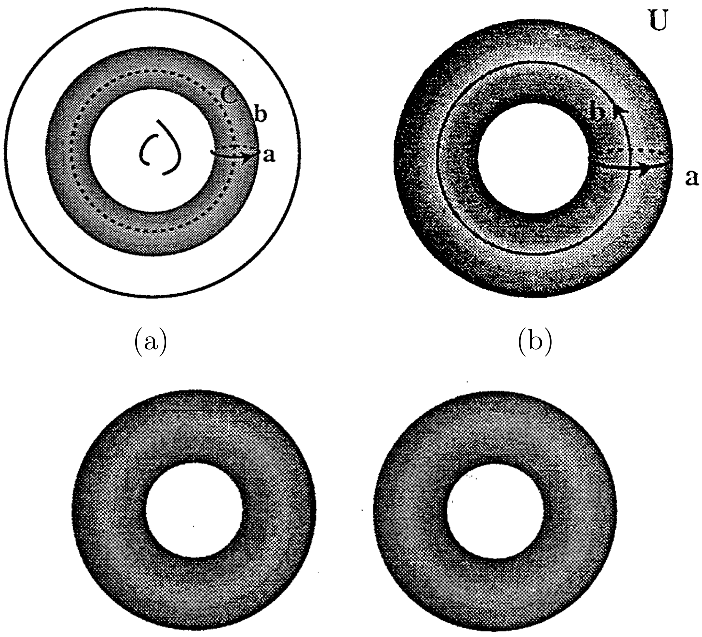
Surgery and S-transformation

We will try to perform a surgery on $\mathbf{S}^2\times\mathbf{S}^1$ to make $\mathbf{S}^3$. The procedure, as usual, begins with picking a circle C in $\mathbf{S}^2\times\mathbf{S}^1$ as depicted in Fig. 28a. We then remove the solid torus $\mathbf{U}$ (the tubular neighbourhood of C) from $\mathbf{S}^2\times\mathbf{S}^1$. Notice that a solid torus $\mathbf{U}$ (Fig. 28b) embedded in $\mathbb{R}^3$, has a torus $\mathbf{T}^2$ as its boundary. Moreover, the torus $\mathbf{T}^2$ is invariant under inversion, so that its exterior (including the point at infinity) is a second solid torus $\mathbf{U}'$. To be precise, $\mathbf{U}'$ is a translate of $\mathbf{U}$ in Euclidean space.

We will now have two identical solid tori, as depicted in Fig. 28c. Since a solid torus $\mathbf{U}$ is $\mathbf{D}\times\mathbf{S}^1$ ($\mathbf{D}$ a 2-dimensional disc) and $\mathbf{U}'$ is $\mathbf{D}'\times\mathbf{S}^1$, when we glue them along their boundary with the identity map, we obtain $\mathbf{S}^2\times\mathbf{S}^1$. This follows from the fact that the two discs $\mathbf{D}$ and $\mathbf{D}'$ glued on their boundary make $\mathbf{S}^2$.

This surgery shows that we get back $\mathbf{S}^2\times\mathbf{S}^1$, but it is not what we wanted. If we do the gluing, however, after performing an S-**transformation** on the $\mathbf{T}^2$, the resulting manifold will be instead $\mathbf{S}^3$. The complement of a solid torus $\mathbf{U}$ inside $\mathbf{S}^3$ is indeed another solid torus $\mathbf{U}'$ whose non-contractible cycle b is homologous to the contractible cycle a in the first torus. The action of S-transformation on this homology basis exchanges the 1-cycles a and b of $\mathbf{T}^2$. Specifically it is given by a matrix $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ that maps a to $-b$ and b to a.

The action of S-transformation can be lifted to the physical Hilbert space of the torus $\mathcal{H}_{\mathbf{T}^2}$ (we will discuss in detail in Section 5.7). For $G=SU(2)$, in the Verlinde



(a) (b)

(c)

Figure 28: (a) A surgery is performed on a circle C in $\mathbf{S}^2 \times \mathbf{S}^1$. (b) Sitting in $\mathbb{R}^3$, a torus $\mathbf{T}^2$, with its interior, makes up a solid torus $\mathbf{U}$ that has a contractible cycle a and a non-contractible cycle b. (c) A pair of identical solid tori.

basis or more precisely, the basis of the characters of integrable representations of affine Lie algebra, the matrix elements S_{ij} are given very explicitly as

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\Big(\frac{(i+1)(j+1)\pi}{k+2}\Big). \tag{5.39}$$

Partition Function of $\mathbf{S}^3$

Deducing from the surgery relation (5.30), we have a relation between the partition function of $\mathbf{S}^3$ and that of $\mathbf{S}^2 \times \mathbf{S}^1$,

$$Z(\mathbf{S}^3) = \sum_j S_0{}^j Z(\mathbf{S}^2 \times \mathbf{S}^1; \mathcal{R}_j) \tag{5.40}$$

We have learned from (5.36) that $Z(\mathbf{S}^2 \times \mathbf{S}^1; \mathcal{R}_j)$ is 1 for $j = 0$ and 0 otherwise, so

$$Z(\mathbf{S}^3) = S_{00}. \tag{5.41}$$

For $G = SU(2)$, the value of S_{00} can be determined from (5.39) and we get

$$Z(\mathbf{S}^3) = \sqrt{\frac{2}{k+2}} \sin\Big(\frac{\pi}{k+2}\Big). \tag{5.42}$$

5.5 Framing Dependence of Three-manifolds

We have built $\mathbf{S}^3$ from $\mathbf{S}^2 \times \mathbf{S}^1$, by performing surgery on a certain circle C, using the modular transformation $S : \tau \to -1/\tau$, where τ parametrises a torus $\mathbf{T}^2$. Apart from S, there are other modular transformations that could be used to get $\mathbf{S}^3$ by surgery on the same circle C in $\mathbf{S}^2 \times \mathbf{S}^1$. The general choice would be $T^n S T^m$, with n and m being arbitrary integers, and T being the modular transformation $T : \tau \to \tau + 1$. In fact, S transformation, together with the T transformation ($T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$), generate the modular group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$. The $PSL(2, \mathbb{Z})$ transformation is a special class of homeomorphism of a torus.

Had we used $T^n S T^m$, we would have obtained not (5.41) but

$$Z(\mathbf{S}^3) = (T^n S T^m)_{00}. \tag{5.43}$$

In the Verlinde basis v_i, T is a diagonal matrix with $T \cdot v_i = \exp\left(2\pi i (h_{\mathcal{R}_i} - c/24)\right) \cdot v_i$, where $h_{\mathcal{R}_i}$ is the conformal weight of WZW primary field in the $\mathcal{R}_i$ representation and c is the central charge for affine Lie algebra[17] with symmetry group G at level k. For $G = SU(N)$, $c = \frac{k \dim G}{k + h^\vee} = \frac{k(N^2-1)}{k+N}$. Since $h_{\mathcal{R}_0} = 0$, if we replace (5.41) by (5.43), the partition function transforms as

$$Z \to Z \cdot \exp\left(2\pi i (n - m) \cdot c/24\right). \tag{5.44}$$

This transformation is obtained in a particular surgery, giving $\mathbf{S}^3$ from $\mathbf{S}^2 \times \mathbf{S}^1$, in the canonical framing.

In a general 3-manifold $\mathcal{M}$, one considers the surgery on the same circle C, being determined by an $PSL(2, \mathbb{Z})$ element $u \cdot T^m$. This would have the same effect topologically, but the partition function will contain an extra phase $\exp\left(-2\pi i m \cdot c/24\right)$. Hence two surgeries that have the same effect on the topology of a 3-manifold may have different effects on the framing.

Witten considered *framed 3-manifolds* rather than garden-variety 3-manifolds when he resolved phase ambiguities in the CS partition function, so that it gives invariants of links and manifolds upon regularising the CS theory [11]. A **framed 3-manifold** is a compact oriented 3-manifold equipped with framings. Recall that for a framed link $\mathcal{L}$, it means that we fix a trivialisation of its normal bundle in the 3-manifold $\mathcal{M}$. For a framed 3-manifold $\mathcal{M}$ means that we fix a trivialisation of the tangent bundle.

To be precise, a **framing** (also called the **2-framing**) of a 3-manifold $\mathcal{M}$ is a homotopy class of trivialisations of the tangent bundle $2T\mathcal{M} = T\mathcal{M} \oplus T\mathcal{M}$ (as a Spin(6) bundle)[18]. Two framings are said to be equivalent if they are ambient isotopic after being included diagonally into the frame bundle $F(T\mathcal{M} \oplus T\mathcal{M})$. This

[17]The affine Lie algebra is also known by the term current algebra in Witten (1989) [11].

[18]The tangent bundle $2T\mathcal{M}$ has a natural spin structure arising from the lift to Spin(6) of the diagonal embedding $SO(3) \hookrightarrow SO(3) \times SO(3) \hookrightarrow SO(6)$. Since $T\mathcal{M}$ is trivial, so is $2T\mathcal{M}$.

equivalence relation sees $\pi_3(SO(3))$ but not $\pi_1(SO(3))$, since the diagonal inclusion $SO(3) \hookrightarrow SO(3) \oplus SO(3) \subset SO(6)$ is zero on π_1.

From another point of view, the framings are the sections of the frame bundle $F(T\mathcal{M})$. To see why, we define a framing of $\mathcal{M}$ to be a real singular cycle in the singular 3-chain group $C_3(F(T\mathcal{M}), \partial F(T\mathcal{M}); \mathbb{R})$ which projects to a representative of the fundamental class in the third homology group $H_3(\mathcal{M}, \partial\mathcal{M}; \mathbb{R})$. A section of $F(T\mathcal{M})$ clearly gives rise to such a cycle. Framings α and β are considered equivalent if $\alpha - \beta$ is a boundary (null-homologous cycle) in $C_3(F(T\mathcal{M}); \mathbb{R})$. Notice that in order for $\alpha - \beta$ to be a cycle we must have $\partial\alpha = \partial\beta$. A detailed explanation on 2-framings can be found in Atiyah (1990) [51].

5.6 Diagonalisation of Fusion Rules

We have found that the partition function of $\mathbf{S}^3$ is the (0,0)-component of the S-transformation, $Z(\mathbf{S}^3) = S_{00}$. We can go further and add an unknotted Wilson loop on $\mathbf{S}^3$ in an arbitrary representation $\mathcal{R}_j$ and determine its partition function $Z(\mathbf{S}^3; \mathcal{R}_j)$. To do this, we start on $\mathbf{S}^2 \times \mathbf{S}^1$ with a Wilson loop in the $\mathcal{R}_j$ representation running parallel to the circle C on which we are doing surgery, as indicated in Fig. 29a. Carrying out the same surgery as before turns $\mathbf{S}^2 \times \mathbf{S}^1$ into $\mathbf{S}^3$, with a Wilson loop in the $\mathcal{R}_j$ representation on $\mathbf{S}^3$. Application of (5.30) now gives

$$Z\left(\mathbf{S}^3; R_j\right) = \sum_i S_0{}^i Z\left(\mathbf{S}^2 \times \mathbf{S}^1; R_i, R_j\right) \tag{5.45}$$

The right-hand side of this equation can be evaluated using (5.37). The partition function for a Wilson loop in an arbitrary R_j representation is then

$$Z\left(\mathbf{S}^3; \mathcal{R}_j\right) = \sum_i S_0{}^i g_{ij} = S_{0j}. \tag{5.46}$$

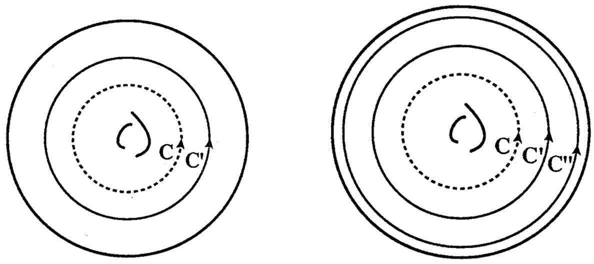
If we represent the correlation function $W_{\mathcal{R}}$ as a ratio $W_{\mathcal{R}} = Z\left(\mathbf{S}^3; \mathcal{R}\right) / Z\left(\mathbf{S}^3\right)$, as in (5.20), and take $G = SU(2)$ with $\mathcal{R} = \square$, the fundamental representation of $SU(2)$, then the formula (5.39) gives

$$W_{\square} = \frac{S_{01}}{S_{00}} = \frac{\sin(2\pi/(k+2))}{\sin(\pi/(k+2))} \tag{5.47}$$

It is easy to check that setting $N = 2$ in (5.25) gives the same equation.

Taking this one step further, we can calculate by the same methods the partition function $Z\left(\mathbf{S}^3; \mathcal{R}_j, \mathcal{R}_k\right)$ for $\mathbf{S}^3$ with two unknotted, unlinked Wilson loops in representations $\mathcal{R}_j$ and $\mathcal{R}_k$. As in Fig. 29b we start on $\mathbf{S}^2 \times \mathbf{S}^1$ with two Wilson loops in representations $\mathcal{R}_j$ and $\mathcal{R}_k$, parallel to the circle C on which surgery is to be performed. Carrying out the surgery, we get to $\mathbf{S}^3$ with the desired unlinked, unknotted circles. The surgery relation (5.30) tells us that

$$Z\left(\mathbf{S}^3; \mathcal{R}_j, \mathcal{R}_k\right) = \sum_i S_0{}^i Z\left(\mathbf{S}^2 \times \mathbf{S}^1; \mathcal{R}_i, \mathcal{R}_j, \mathcal{R}_k\right) \tag{5.48}$$



(a) (b)

Figure 29: In (a), in parallel to the circle C on which we perform surgery, there is a circle C' on which we place a Wilson loop in the $\mathcal{R}_j$ representation. In (b), there are two parallel circles C' and C'' with Wilson loops in representations $\mathcal{R}_j$ and $\mathcal{R}_k$.

The right-hand side can be evaluated with (5.38), while the left-hand side can be reduced to (5.46) using (5.22). We get

$$\frac{S_{0j}S_{0k}}{S_{00}} = \sum_i S_0{}^i N_{ijk} \tag{5.49}$$

This equation is a special case of a celebrated conjecture by Erik Verlinde,

$$\frac{S_{ij}S_{ik}}{S_{0i}} = \sum_l S_i{}^l N_{ljk}, \tag{5.50}$$

which has been proved by Moore and Seiberg [49] and Witten [11]. This equation is given the name **Verlinde formula** and is one of the most important results in CFT. It is interpreted as in Verlinde's statement that "the modular transformation S diagonalises the fusion rules" [8].

To verify the statement, we notice that in the Verlinde basis v_i, the structure constants of the Verlinde algebra are by definition $v_i v_j = \sum_k N_{ij}{}^k v_k$, where $N_{ij}{}^k = \sum_l N_{ijl} g^{lk}$. If we introduce a new basis $w_i = S_{0i} \cdot \sum_r S_i{}^r v_r$, then

$$w_i w_j = \sum_{k,l} S_i{}^k S_j{}^l v_k v_l \cdot S_{0i} S_{0j}. \tag{5.51}$$

Using $v_k v_l = \sum_m N_{kl}{}^m v_m$ and the Verlinde formula (5.50), this becomes

$$w_i w_j = \sum_{l,m} S_j{}^l v_m \cdot S_{il} S_i{}^m \cdot S_{0j} \tag{5.52}$$

Using the unitarity of S, in the form $\sum_l S_j{}^l S_{il} = \delta_{ij}$, we see that

$$w_i w_j = \delta_{ij} \sum_m S_j{}^m v_m \cdot S_{0j} = \delta_{ij} \cdot w_j \tag{5.53}$$

showing that the Verlinde algebra has been diagonalised and that the w_i are idempotents.

The outstanding importance of the Verlinde formula can be seen in the fact that it combines local as well as global properties in CFT: the fusion numbers N_{ijk} contain information about the local OPE of two fields whereas the modular transformation S is related to the global modular invariance of partition functions on the torus!

5.7 Knots and Links Revisited

In this section, we will look at how computation of the vevs of some knots and links in $\mathbf{S}^3$ are done using the theory of affine Lie algebra. We denote a state in the Hilbert space of torus $\mathcal{H}_{\mathbf{T}^2}$ as $|\mathcal{R}\rangle$ in which $|\mathcal{R}\rangle = |\Psi_{\mathbf{U},W_\mathcal{R}}(\mathcal{A})\rangle$, the wavefunction given by the CS path integral on the solid torus $\mathbf{U}$ with the insertion of a Wilson loop operator $W_\mathcal{R}$ in the representation $\mathcal{R}$ around the non-contractible cycle:

$$\Psi_{\mathbf{U},W_\mathcal{R}}(\mathcal{A}) = \langle\mathcal{A}|\Psi_{\mathbf{U},W_\mathcal{R}}\rangle = \int_{A|_\Sigma=\mathcal{A}} DA\, e^{iS}\, W_\mathcal{R}. \tag{5.54}$$

In particular, the path integral over $\mathbf{U}$ with no operator insertion gives $|0\rangle$, the "vacuum" state.

The partition function on a 3-manifold $\mathcal{M}$ is given by $Z(\mathcal{M}) = \langle\Psi_{\mathcal{M}_2}|\, U_f\, |\Psi_{\mathcal{M}_1}\rangle$ where U_f is an operator acting on the Hilbert space $\mathcal{H}_\Sigma$ of the boundary Σ and is a representation of the diffeomorphism $f : \Sigma \to \Sigma$. The 3-manifold $\mathcal{M}$ is a connected sum of the two 3-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ glued together by the homeomorphism f, i.e. $\mathcal{M} = \mathcal{M}_1 \cup_f \mathcal{M}_2$.

As mentioned before, if we perform a surgery on two solid tori by gluing them along their boundary Σ (i.e. a torus $\mathbf{T}^2$) with the identity map, we obtain $\mathbf{S}^2 \times \mathbf{S}^1$. The partition function is then

$$Z(\mathbf{S}^2 \times \mathbf{S}^1) = \langle 0|0\rangle = 1, \tag{5.55}$$

which is what we had in (5.33). If we perform an S-transformation on the boundary $\mathbf{T}^2$ before gluing the two solid tori, we get $\mathbf{S}^3$ with its partition function

$$Z(\mathbf{S}^3) = \langle 0|S|0\rangle = S_{00}. \tag{5.56}$$

Some Notions of Affine Lie Algebra

To evaluate S_{00}, instead of using the formula in (5.39), we now turn to the more complicated theory of affine lie algebra which applies for any Lie group G. Recall that $\mathcal{H}_\Sigma$ is the space of conformal blocks of a WZW model on the boundary Σ with gauge group G and level k. For $\Sigma = \mathbf{S}^2$, the space of conformal blocks is one-dimensional, so $\mathcal{H}_{\mathbf{S}^2}$ is spanned by a single element. For $\Sigma = \mathbf{T}^2$, the space of conformal blocks is in one-to-one correspondence with the integrable representations of the affine Lie algebra associated to G at level k. The states $|\mathcal{R}\rangle$ in $\mathcal{H}_{\mathbf{T}^2}$ are the integrable representations of the WZW model at level k.

It is instructive to introduce the following notations that we will use. The fundamental weights of G will be denoted by λ_i, and the simple roots by $\alpha_i, i = 1, \cdots, r$, where r denotes the rank of G. $|\Delta_+|$ denotes the number of positive roots of the root system Δ, given by $(|G| - r)/2$, where $|G|$ is the order of G. The weight and root lattices of G are denoted, respectively, by Λ^w and Λ^r. The fundamental chamber $\mathcal{F}_l$ of the weight lattice of G is given by $\Lambda^w / l\Lambda^r$, where $l = k + h^\vee$, modded out by the action of the Weyl group. For example, in $SU(N)$ a weight $p = \sum_{i=1}^r p_i \lambda_i$ is in $\mathcal{F}_l$ if

$$\sum_{i=1}^{r} p_i < l \quad \text{and} \quad p_i > 0, \quad i = 1, \cdots, r. \tag{5.57}$$

A representation given by a highest weight Λ is integrable if $\rho + \Lambda$ is in the fundamental chamber $\mathcal{F}_l$. Here ρ denotes the Weyl vector, given by the sum of the fundamental weights or the half-sum of the positive roots. The state $|\mathcal{R}\rangle$ in $\mathcal{H}_{\mathbf{T}^2}$ is also denoted by $|p\rangle = |\rho + \Lambda\rangle$, where $\rho + \Lambda \in \mathcal{F}_l$, as we have stated, is an integrable representation of the WZW model at level k. The state $|\rho\rangle$ will be denoted by $|0\rangle$. The states $|\mathcal{R}\rangle$ can be chosen to be orthonormal, i.e. $\langle \mathcal{R}|\mathcal{R}'\rangle = \delta_{\mathcal{R}\mathcal{R}'}$ [11, 52].

Explicit Calculations

The elements of modular transformation matrix S in the basis of integrable representations is given by

$$S_{pp'} = \frac{i^{|\Delta_+|}}{(k + h^\vee)^{r/2}} \left(\frac{\mathrm{Vol}\, \Lambda^w}{\mathrm{Vol}\, \Lambda^r} \right)^{\frac{1}{2}} \sum_{w \in \mathcal{W}} \epsilon(w) \exp \left(-\frac{2\pi i}{k + h^\vee} p \cdot w\,(p') \right). \tag{5.58}$$

In this equation, $\epsilon(w)$ is the determinant of the action of w on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This is equal to $(-1)^{\ell(w)}$, where $\ell(w)$ is the length of the Weyl group element w, defined to be the minimal number of reflections with respect to simple roots such that w equals the product of those reflections. The volume of the weight (root) lattice is denoted by $\mathrm{Vol}\, \Lambda^w$ ($\mathrm{Vol}\, \Lambda^r$). Often, $S_{\mathcal{R}\mathcal{R}'}$ is written for $S_{pp'}$, where $p = \Lambda + \rho$, $p' = \Lambda' + \rho$ and Λ, Λ' are the highest weights corresponding to the representations $\mathcal{R}, \mathcal{R}'$.

In the case of $G = SU(2)$, we have $\Lambda + \rho = \frac{a}{2}(1, -1) + \frac{1}{2}(1, -1)$ and the Weyl reflection w exchanges 1 and -1. We will then obtain

$$\begin{aligned} S_{ab} &= \frac{i}{(k+2)^{1/2}} \left(\frac{1}{2} \right)^{1/2} \left[\exp \left(\frac{2\pi i}{k+2} \frac{(a+1)(b+1)}{2} \right) - \exp \left(-\frac{2\pi i}{k+2} \frac{(a+1)(b+1)}{2} \right) \right] \\ &= \sqrt{\frac{2}{k+2}} \sin \left(\frac{(a+1)(b+1)\pi}{k+2} \right), \end{aligned} \tag{5.59}$$

which is exactly equation (5.39).

By using Weyl denominator formula,

$$\sum_{w\in\mathcal{W}} \epsilon(w)\ \mathrm{e}^{w(\rho)} = \prod_{\alpha\in\Delta_+} 2\sinh\frac{\alpha}{2}, \tag{5.60}$$

where $\alpha > 0$ are positive roots, one finds

$$Z(\mathbf{S}^3) = S_{00} = \frac{1}{(k+h^\vee)^{r/2}} \left(\frac{\mathrm{Vol}\,\Lambda^w}{\mathrm{Vol}\,\Lambda^r}\right)^{\frac{1}{2}} \prod_{\alpha\in\Delta_+} 2\sin\left(\frac{\pi(\alpha\cdot\rho)}{k+h^\vee}\right). \tag{5.61}$$

The above result can be generalised in order to compute correlation functions in $\mathbf{S}^3$ with some knots and links. Consider a solid torus $\mathbf{U}$ where a Wilson loop in representation $\mathcal{R}$ has been inserted. The corresponding state is $|\mathcal{R}\rangle$, as we explained before. If we now glue this to an empty solid torus after an S-transformation, we will obtain an unknot in $\mathbf{S}^3$. The partition function with the insertion is then

$$Z_{\mathcal{R}} = \langle 0|S|\mathcal{R}\rangle = S_{0\mathcal{R}}. \tag{5.62}$$

It follows that the correlation function for the unknot in $\mathbf{S}^3$, in representation $\mathcal{R}$, is given by

$$W_{\mathcal{R}}(\text{unknot}) = \frac{Z_{\mathcal{R}}}{Z(\mathbf{S}^3)} = \frac{S_{0\mathcal{R}}}{S_{00}} = \frac{\sum_{w\in\mathcal{W}} \epsilon(w)\,\mathrm{e}^{-\frac{2\pi i}{k+h^\vee}\rho\cdot w(\Lambda+\rho)}}{\sum_{w\in\mathcal{W}} \epsilon(w)\,\mathrm{e}^{-\frac{2\pi i}{k+h^\vee}\rho\cdot w(\rho)}}. \tag{5.63}$$

This expression can be written in terms of characters of the group G. By using Weyl character formula[19]:

$$\mathrm{ch}_\Lambda(a) = \frac{\sum_{w\in\mathcal{W}} \epsilon(w)\,\mathrm{e}^{a\cdot w(\Lambda+\rho)}}{\sum_{w\in\mathcal{W}} \epsilon(w)\,\mathrm{e}^{a\cdot w(\rho)}}, \tag{5.64}$$

we can write

$$W_{\mathcal{R}}(\text{unknot}) = \mathrm{ch}_{\mathcal{R}}\left(-\frac{2\pi i}{k+h^\vee}\rho\right). \tag{5.65}$$

Moreover, using Weyl denominator formula we finally obtain

$$W_{\mathcal{R}}(\text{unknot}) = \prod_{\alpha\in\Delta_+} \frac{\sin\left(\frac{\pi}{k+h^\vee}\alpha\cdot(\Lambda+\rho)\right)}{\sin\left(\frac{\pi}{k+h^\vee}\alpha\cdot\rho\right)}. \tag{5.66}$$

This quantity is often called the **quantum dimension** of $\mathcal{R}$, denoted by $\dim_q \mathcal{R}$.

We can also consider a solid torus with a Wilson loop in representation $\mathcal{R}$ glued to another solid torus with the representation $\mathcal{R}'$ via an S-transformation. What we obtain is clearly a link in $\mathbf{S}^3$ with two components, which is the right-handed Hopf link with linking number $+1$. The partition function with this insertion is:

$$Z_{\mathcal{R}\mathcal{R}'} = \langle \mathcal{R}'|S|\mathcal{R}\rangle, \tag{5.67}$$

[19] The character of the representation $\mathcal{R}$ evaluated on an element $a \in \Lambda^w \otimes \mathbb{R}$ is defined by $\mathrm{ch}_{\mathcal{R}}(a) = \sum_{\mu\in M_{\mathcal{R}}} \mathrm{e}^{a\cdot\mu}$, where $M_{\mathcal{R}}$ is the set of weights associated to the irreducible representation $\mathcal{R}$.

so the correlation function is

$$W_{\mathcal{R}\mathcal{R}'} \equiv W_{\mathcal{R}\mathcal{R}'}\left(\mathrm{Hopf}^{+1}\right) = \frac{S_{\overline{\mathcal{R}}'\mathcal{R}}}{S_{00}} = \frac{S^{-1}_{\mathcal{R}'\mathcal{R}}}{S_{00}}, \tag{5.68}$$

where the superscript $+1$ refers to the linking number. Here, we have used that the bras $\langle\mathcal{R}|$ are canonically associated to conjugate representations $\overline{\mathcal{R}}$, and that $S_{\overline{\mathcal{R}}'\mathcal{R}} = S^{-1}_{\mathcal{R}'\mathcal{R}}$. The invariant of the left-handed Hopf link with linking number -1 can be obtained by noticing that the two Hopf links are related by changing the orientation of one of the components. We then have

$$W_{\mathcal{R}\mathcal{R}'}\left(\mathrm{Hopf}^{-1}\right) = \frac{S_{\mathcal{R}'\mathcal{R}}}{S_{00}} \tag{5.69}$$

using the property that $\mathrm{Tr}_{\mathcal{R}}\, U_{\mathcal{K}^{-1}} = \mathrm{Tr}_{\mathcal{R}}\, U^{-1}_{\mathcal{K}} = \mathrm{Tr}_{\overline{\mathcal{R}}}\, U_{\mathcal{K}}$. When we take $G = U(N)$, the above vevs for unknots and Hopf links can be evaluated explicitly in terms of **Schur polynomials**, which gives the character of the unitary group in the representation $\mathcal{R}$. The results were shown to be related to the HOMFLY polynomial in Morton and Lukac (2003) [53].

One may realise that although CS theory is exactly solvable, in practice the computation of vevs for knots and links can be complicated. The procedure of getting the skein relation becomes cumbersome if the number of steps becomes very large. Moreover, although one can write down skein relations for arbitrary representations, they do not determine uniquely the value of the invariant, and hence other techniques are needed.

A particularly useful framework to compute knot invariants is the formalism of **knot operators** (Labastida *et al.*, 1991 [54]). In this formalism, one constructs operators that "create" knots wrapped around a Riemann surface Σ in the representation $\mathcal{R}$ of the gauge group associated to the highest weight Λ:

$$W^{\mathcal{K}}_{\Lambda} : \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma). \tag{5.70}$$

The topology of Σ restricts the type of knots that one can consider. So far, these operators have been constructed only in the case when $\Sigma = \mathbf{T}^2$, which we are going to discuss next.

5.8 Torus Knots

We are going to discuss the computation of the correlation functions of knots that are put on the surface of a torus, known as **torus knots**. They are labelled by two coprime integers (n, m) that specify the number of times that they wrap the two cycles of the torus. Here, n refers to the winding number around the non-contractible cycle of the solid torus $\mathbf{U}$, while m refers to the contractible one. For instance, the trefoil knot 3_1 is the $(2, 3)$ torus knot, and the knot 5_1 is the $(2, 5)$ torus knot. The operator that creates the (n,m) torus knot in the representation associated to Λ will be denoted by $W_\Lambda^{(n,m)}$, and it has a fairly explicit expression for $G = SU(N)$:

$$W_\Lambda^{(n,m)}|p\rangle = \mathrm{e}^{2\pi i n m h_{\rho+\Lambda}} \sum_{\mu\in M_\Lambda} \exp\left(i\pi\mu^2 \frac{nm}{k+N} + 2\pi i \frac{m}{k+N} p\cdot\mu\right)|p+n\mu\rangle. \quad (5.71)$$

Here, $|p\rangle$ is an arbitrary state in $\mathcal{H}_{\mathbf{T}^2}$, and M_Λ is the set of weights corresponding to the irreducible representation with highest weight Λ. The factor involving the conformal weight $h_{\rho+\Lambda}$ is introduced in order to obtain the invariant in the canonical framing.

We compute the vev of the Wilson loop around a torus knot in $\mathbf{S}^3$ in a similar surgery as we have done before. First, we make a Heegaard splitting of $\mathbf{S}^3$ into two solid tori. Then, we put the torus knot on the surface of one of the solid tori by acting with the knot operator (5.71) on the vacuum state $|\rho\rangle$. Finally, we glue together the tori by performing an S-transformation. The correlation function of the Wilson loop is then given by:

$$\langle W_\Lambda^{(n,m)}\rangle = \frac{\langle\rho|S W_\Lambda^{(n,m)}|\rho\rangle}{\langle\rho|S|\rho\rangle}. \quad (5.72)$$

There is also a more compact way to write (5.72). When the operator $W_\Lambda^{(n,m)}$ acts on the vacuum state $|\rho\rangle$, the right-hand side of (5.71) is a linear combination of states of the form $|\rho+n\mu\rangle$, where $\mu\in M_\Lambda$. The corresponding weights have representatives in the Weyl alcove[20] $\mathcal{A}_l$ that can be obtained by a series of Weyl reflections. In other words, given n and μ, there is a weight $\rho+\xi$ in $\mathcal{A}_l$ and a Weyl reflection $w_\xi\in\mathcal{W}$ such that $\rho+n\mu = w_\xi(\rho+\xi)$. We will denote the set of representatives of the weights $\rho+n\mu$ in $\mathcal{A}_l$ by $\mathcal{M}(n,\Lambda)$. Then, the CS invariant of a torus knot (n,m) can be written as:

$$\mathrm{e}^{2\pi i n m h_{\rho+\Lambda}} \sum_{\rho+\xi\in\mathcal{M}(n,\Lambda)} \epsilon(w_\xi)\exp\left(\frac{i\pi m}{n(k+N)}\xi\cdot(\xi+2\rho)\right)\mathrm{ch}_\xi\left(-\frac{2\pi i}{k+N}\rho\right). \quad (5.73)$$

Since the representatives $\rho+\xi$ live in $\mathcal{A}_l$, the weights ξ can be considered as highest weights for a representation, hence the character in (5.73) makes sense.

As an example, we compute the invariant of a torus knot (n,m) in the fundamental representation, where $\Lambda=\lambda_1$. By performing Weyl reflections, $\mathcal{M}(n,\lambda_1)$ is given by the following weights [55]:

$$\rho+(n-i)\lambda_1+\lambda_i, \quad i=1,\cdots,N. \quad (5.74)$$

[20] Weyl alcove is a subset of the Weyl chamber. In fact, each Weyl chamber is a union of alcoves. Sharing a similar construction with the Weyl chambers – the connected components of an open set obtained by removing the finitely many hyperplanes from a finite dimensional real Euclidean space V – removing all hyperplanes in H (a collection of all affine hyperplanes) from V leaves an open set whose connected components are called the Weyl alcoves.

The characters are just the quantum dimensions of the weights (5.74), and skipping the details we have the following result:

$$
\begin{aligned}
W_{\square}^{(n,m)} =& q^{\frac{1}{2}}\lambda^{-\frac{1}{2}} \frac{(\lambda q^{-1})^{\frac{(m-1)(n-1)}{2}}}{q^n - 1} \\
& \times \sum_{\substack{p+i+1=n \\ p,\, i\geq 0}} (-1)^i q^{mi+\frac{1}{4}(p(p+1)-i(i+1))} \frac{\prod_{j=-p}^{i}(\lambda - q^j)}{[i]![p]!},
\end{aligned}
\tag{5.75}
$$

where $[x]! = [x][x-1]\cdots[1]$ are the *q-factorials* with $[x] = q^{\frac{x}{2}} - q^{-\frac{x}{2}}$ being the *q-numbers*. If we divide by the vev of the unknot, we find the expression for the HOMFLY polynomial first obtained by Jones (1987) [56]. For instance, one has, for the trefoil:

$$
W_{\square}^{(2,3)} = \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}(-2\lambda^{\frac{1}{2}} + 3\lambda^{\frac{3}{2}} - \lambda^{\frac{5}{2}}) + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(-\lambda^{\frac{1}{2}} + \lambda^{\frac{3}{2}}). \tag{5.76}
$$

With more effort, one can write down formulae for the invariants of torus knots and links in arbitrary representations, as shown in the works of Labastida *et al.* [55, 57, 58], although they are rather complicated. They afford, however, a systematic computation of the invariants of these knots. For the trefoil in representations with two boxes one finds:

$$
\begin{aligned}
W_{\square\square}^{(2,3)} =& \frac{(\lambda-1)(\lambda q-1)}{\lambda(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(1+q)}\Big((\lambda q^{-1})^2(1 - \lambda q^2 + q^3 \\
& - \lambda q^3 + q^4 - \lambda q^5 + \lambda^2 q^5 + q^6 - \lambda q^6)\Big), \\
W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}^{(2,3)} =& \frac{(\lambda-1)(\lambda-q)}{\lambda(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2(1+q)}\Big((\lambda q^{-2})^2(1 - \lambda - \lambda q \\
& + \lambda^2 q + q^2 + q^3 - \lambda q^3 - \lambda q^4 + q^6)\Big).
\end{aligned}
\tag{5.77}
$$

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