Chern-Simons Theory and Knot Polynomials

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Communications in Mathematical Physics

Quantum Field Theory and the Jones Polynomial *

Edward Witten **

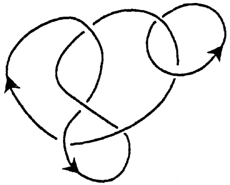
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

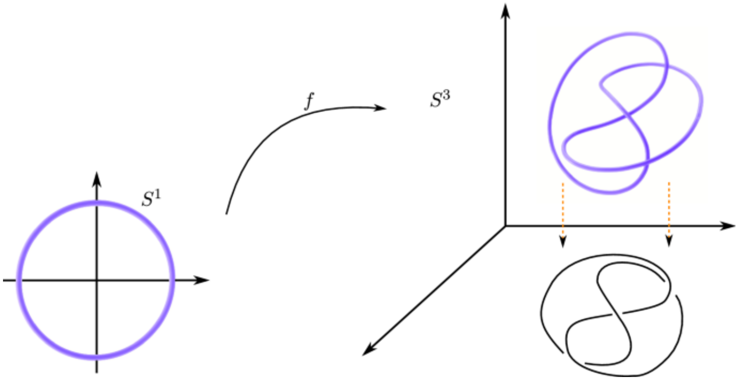
1990 Fields Medalists:
Edward Witten, Shigefumi Mori, Vaughan Jones, Vladimir Drinfeld

Overview

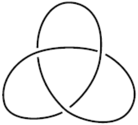
What are knots and links?

- **Knot** $\mathcal{K}$: a smooth embedding of codimension 2 in a 3-manifold $\mathcal{M}$, that is diffeomorphic to S^1.
  Formally, $\mathcal{K}$ is a *simple closed curve* (also known as *Jordan curve*) that is a nearly injective and continuous function $\mathcal{K} : [0,1] \to \mathcal{M}$, with the only "non-injectivity" being $\mathcal{K}(0) = \mathcal{K}(1)$.
- **Link** $\mathcal{L}$: a smooth embedding into a 3-manifold $\mathcal{M}$ that is diffeomorphic to a finite disjoint union of knots: $\mathcal{L} = \mathcal{K}_1 \sqcup \mathcal{K}_2 \sqcup \cdots \sqcup \mathcal{K}_n$.




S^1
f
S^3

Knot & link diagrams



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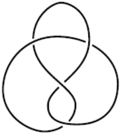

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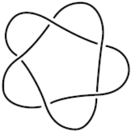

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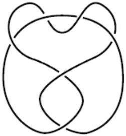

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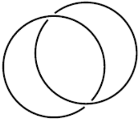
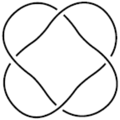


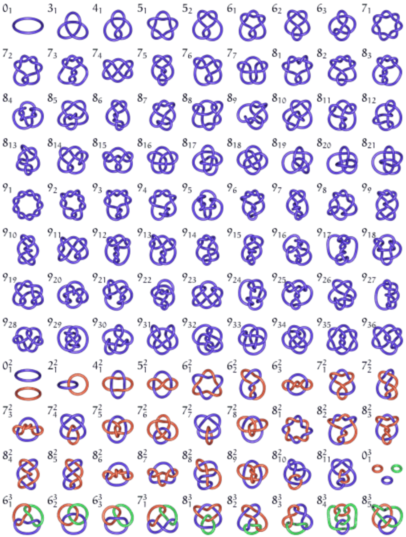
2_1^2



4_1^2

- A **knot/link diagram** D is the projection of a knot/link onto a plane with crossings indicated
- **Standard notation**: x_n^L
  x: crossing number,
  L: number of components (only for links with $L > 1$),
  n: number used to enumerate knots and links in a given set characterized by x and L

Tait knot table

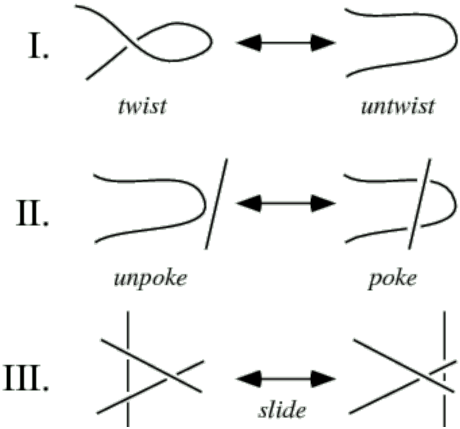


Ambient isotopy

Two links $\mathcal{L}$ and $\mathcal{L}'$ are **ambient isotopic** if there is a smooth map $\alpha : [0,1] \times \mathcal{M} \to \mathcal{M}$ such that for each value $t \in [0,1]$, the map $\alpha(t,\cdot) : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, and $\alpha(0,\cdot)$ is the identity map on $\mathcal{M}$, while $\alpha(1,\cdot)$ maps $\mathcal{L}$ to $\mathcal{L}'$.

Are all the knots the same (ambient isotopic) as the unknot/circle?

Reidemeister moves



- In the 1930s, Reidemeister first rigorously proved that knots exist, which are distinct from the unknot.
- He did this by showing that all knot deformations can be reduced to a sequence of three types of "moves".

Theorem (Reidemeister's theorem)

Two knots/links can be continuously deformed into each other iff any diagram of one can be transformed into a diagram of the other by a sequence of Reidemeister moves.

Reidemeister's theorem guarantees that moves I, II, and III correspond to ambient isotopy of knot/link diagrams.

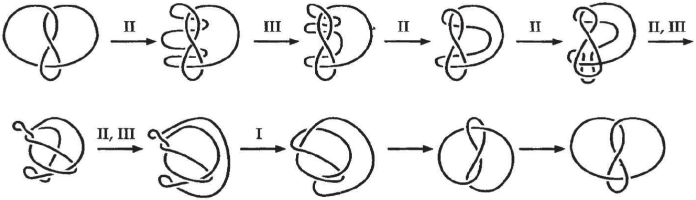


Figure-eight knot is amphichiral.

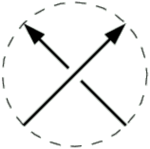
Invariants

- **Linking number**:

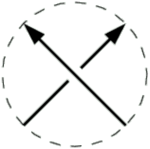
$$lk(\mathcal{K}_1, \mathcal{K}_2) = \frac{1}{2} \sum_{p \in D} \epsilon(p)$$

  for all crossing points p in D with $\epsilon(p) = \pm 1$ being a sign associated to the crossings.

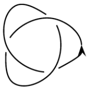
- The linking number of a link $\mathcal{L}$ with components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$ is $lk(\mathcal{L}) = \sum_{\alpha < \beta} lk(\mathcal{K}_\alpha, \mathcal{K}_\beta)$.



$\epsilon(p) = +1$



$\epsilon(p) = -1$



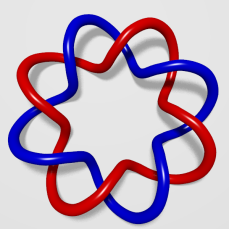
The *left-handed trefoil knot*

The *right-handed trefoil knot*

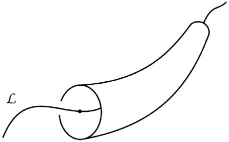
- In terms of the Gauss linking integral,

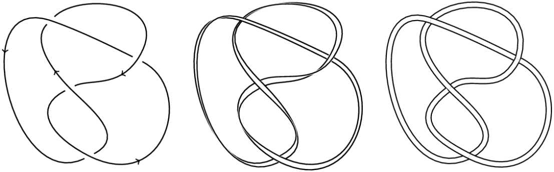
$$lk(\mathcal{L}_\alpha, \mathcal{L}_\beta) = \frac{1}{4\pi} \oint_{\mathcal{L}_\alpha} dx^\mu \oint_{\mathcal{L}_\beta} dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3},$$

  where the distance $|x-y|$ is computed by means of the flat (Euclidean) metric of $\mathcal{M}$.
- This integral is well-defined and finite except near $x = y$.
- $\alpha = \beta$ (self-linking)?



- The **framing** of a link $\mathcal{L} \subset \mathcal{M}$ is a normal vector field n on $\mathcal{L}$, such that $n_p \notin T_p\mathcal{L}$ for all $p \in \mathcal{L}$.
- Displacing $\mathcal{L}$ slightly in the direction of framing, one gets a new link, called a **framed link**, $\mathcal{L}_f$, in the tubular neighbourhood of $\mathcal{L}$ (tubular neighbourhood $\mathcal{L}$ is simply a torus whose core is $\mathcal{L}$).

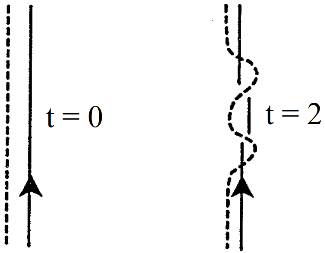




(A) Knot Diagram (B) Framed Knot (C) Blackboard Framing

- **Self-linking number/cotorsion/writhe** is the linking number of a link and its framing, $lk(\mathcal{L}, \mathcal{L}_f)$.
- Also defined as a t-**fold twist** in the framing of $\mathcal{L}$.

$$\phi(\mathcal{L}) = lk(\mathcal{L}, \mathcal{L}_f) = t(\mathcal{L})$$
$$\phi(\mathcal{L}) = \frac{1}{4\pi} \oint_{\mathcal{L}} dx^{\mu} \oint_{\mathcal{L}_f} dy^{\nu} \epsilon_{\mu\nu\rho} \frac{(x-y)^{\rho}}{|x-y|^3}$$



Polynomial invariants

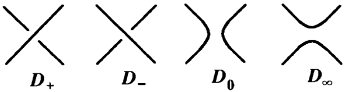
- **Jones polynomial** of an oriented link $\mathcal{L}$, $V(\mathcal{L})$ or $V^{\mathcal{L}}(t)$, is the Laurent polynomial in $t^{1/2}$ with integer coefficients, defined by

$$V^{\mathcal{L}}(t) = (-A)^{-3\phi(D)} \langle D \rangle \Big|_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

- **Laurent polynomial** over a field $\mathbb{F}$:

$$p = \sum_{k \in \mathbb{Z}} p_k t^k, \quad p_k \in \mathbb{F}$$

- **Kauffman bracket** maps a diagram D to $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$, characterised by
  1. $\langle \bigcirc \rangle = 1$
  2. $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^2)\langle D \rangle$
  3. $\langle D_+ \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle$



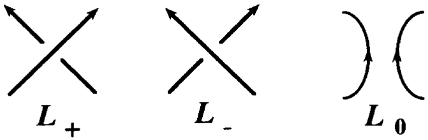
Jones polynomial, $V(\mathcal{L})$ or $V^{\mathcal{L}}(t)$, is a function,

$$V : \{\text{oriented links in } \mathcal{M}\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

which is defined by the axioms:

1. Invariance: $V(\mathcal{L})$ is invariant under ambient isotopy of $\mathcal{L}$, i.e. if $\mathcal{L}_1$ and $\mathcal{L}_2$ are ambient isotopic, then $V(\mathcal{L}_1) = V(\mathcal{L}_2)$.
2. Normalisation: $V(\bigcirc) = 1$.
3. Skein relation: Whenever three oriented links $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{L}_0$ are the same except in the neighbourhood of a point where they differ as in the oriented Conway triple, then we have the skein relation,

$$t^{-1}V(\mathcal{L}_+) - tV(\mathcal{L}_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0.$$

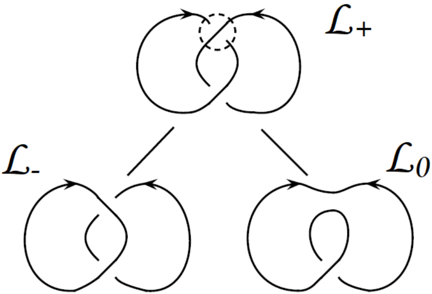


Example: 2 unknots

$\mathcal{L}_-$ $\quad$ $\mathcal{L}_+$ $\quad$ $\mathcal{L}_0$

$$t^{-1}V(\mathcal{L}_+) - tV(\mathcal{L}_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0$$
$$V(\mathcal{L}_+) = 1 = V(\mathcal{L}_-)$$
$$\Rightarrow V(\mathcal{L}_0) = -(t^{1/2} + t^{-1/2})$$

Example: Hopf link



$$t^{-1}V(\mathcal{L}_+) - tV(\mathcal{L}_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0$$
$$t^{-1}V(\mathcal{L}_+) + t(t^{1/2} + t^{-1/2}) + (t^{-1/2} - t^{1/2}) = 0$$
$$\Rightarrow V(\mathcal{L}_+) = -t^{1/2}(1 + t^2)$$

More generally:
HOMFLY(PT) polynomial, $P(\mathcal{L})$ or $P^{\mathcal{L}}(q,\lambda)$, of an oriented link $\mathcal{L}$ is defined by the following three axioms:

1. Invariance: $P(\mathcal{L})$ is invariant under ambient isotopy of $\mathcal{L}$.
2. Normalisation: $P(\bigcirc) = 1$.
3. Skein relation: $q^{-1}P(\mathcal{L}_+) - qP(\mathcal{L}_-) = \lambda P(\mathcal{L}_0)$.

Chern-Simons action

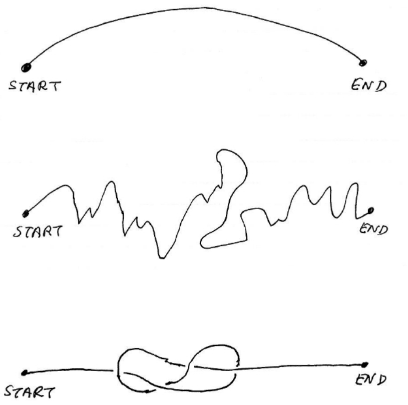
$$
\begin{aligned}
S &= \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\,(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \\
&= \frac{k}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\ \mathrm{Tr}\,(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho)
\end{aligned}
$$

- Compact, oriented three-manifold $\mathcal{M}$ with a compact simple gauge group G
- Non-abelian & topological invariant (no metric)
- k: level (inverse coupling constant) of the CS theory
  ($k \in \mathbb{Z}$ for compact, connected, simple gauge group G)
- Gauge theory: its classical configuration on $\mathcal{M}$ with gauge group G is described by a principal G-bundle over $\mathcal{M}$
- $\frac{\delta S}{\delta A} = 0 \Rightarrow F \equiv dA + A \wedge A = 0 \Rightarrow$ flat G-bundle

Observable

- Holonomy - measure flat connection A? Local observable made of F?
- $\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_n} \rangle = 0$ for a set of operators $\mathcal{O}_{i_1}, \dots, \mathcal{O}_{i_n}$
- Wilson loop around a non-contractible loop:

$$W(A) = \text{Tr Hol}(A) = \text{Tr P}\exp(\oint A)$$

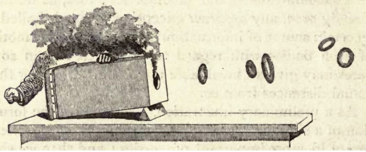


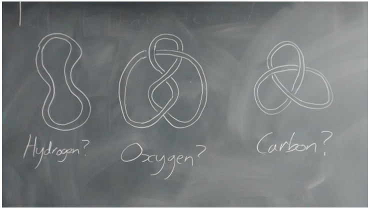
Wilson loop around a knot $\mathcal{K}$:

$$W_{\mathcal{R}}^{\mathcal{K}}(A) = \mathrm{Tr}_{\mathcal{R}}\ \mathrm{P}\exp\left(\oint_{\mathcal{K}} A\right) = \mathrm{Tr}_{\mathcal{R}}\ \mathrm{P}\exp\left(\oint_{\mathcal{K}} A_i\, dx^i\right) \in G$$

"Vortex Atoms", 1867

Lord Kelvin (William Thomson): atoms are knotted vortices in aether





Peter Tait: classification of knots up to 10 crossings

Wilson loop around a knot $\mathcal{K}$:

$$W_{\mathcal{R}}^{\mathcal{K}}(A) = \mathrm{Tr}_{\mathcal{R}}\ \mathrm{P}\exp(\oint_{\mathcal{K}} A) = \mathrm{Tr}_{\mathcal{R}}\ \mathrm{P}\exp(\oint_{\mathcal{K}} A_i\, dx^i) \in G$$

Partition function of a link $\mathcal{L}$:

$$Z_{\mathcal{R}_1 \ldots \mathcal{R}_L}(\mathcal{L}) = \int \mathcal{D}A\, e^{iS}\, (\prod_{\alpha=1}^{L} W_{\mathcal{R}_\alpha}^{\mathcal{K}_\alpha})$$

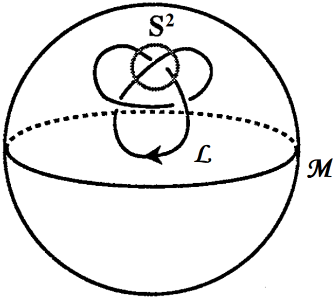
Correlation function of a link $\mathcal{L}$:

$$W_{\mathcal{R}_1 \ldots \mathcal{R}_L}(\mathcal{L}) =< W_{\mathcal{R}_1}^{\mathcal{K}_1} \ldots W_{\mathcal{R}_L}^{\mathcal{K}_L} >= \frac{1}{Z(\mathcal{M})} \int \mathcal{D}A\, e^{iS}\, (\prod_{\alpha=1}^{L} W_{\mathcal{R}_\alpha}^{\mathcal{K}_\alpha}),$$

$$Z(\mathcal{M}) = \int [\mathcal{D}A]\, e^{iS}$$

Jones Polynomial from CS Theory

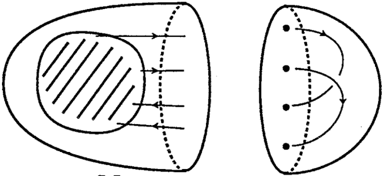
- $\mathcal{M} = S^3$ with a link $\mathcal{L}$ embedded in it
- Wilson loop in the fundamental representation ($\mathcal{R} = \square$) of $G = SU(2)$



Goal: find the skein relation of Jones polynomial for $\mathcal{L}$,

$$t^{-1}V(\mathcal{L}_+) - tV(\mathcal{L}_-) + (t^{-1/2} - t^{1/2})V(\mathcal{L}_0) = 0$$

Step I: Surgery – cut the manifold



- draw a small sphere S^2 about a crossing
- cutting the sphere, we perform a **Heegaard splitting** on $\mathcal{M}$
- simple piece $\mathcal{M}_R$ (interior of S^2) & complicated piece $\mathcal{M}_L$ (exterior of S^2 with complicated details)
- boundaries $\partial\mathcal{M}_R = \partial\mathcal{M}_L = S^2$, of opposite orientations
- on $\partial\mathcal{M}_R$: 4 **marked points** indicating the intersections of $\mathcal{L}$ with S^2, connected by two lines in the interior of the 3-ball

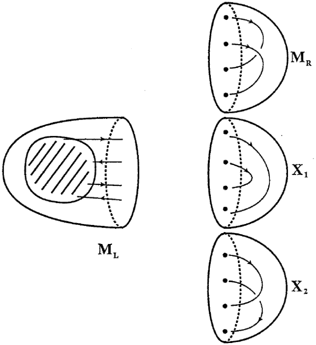
Step II: Quantisation

- associate 2-dimensional physical Hilbert spaces $\mathcal{H}_R$ and $\mathcal{H}_L$ (canonically dual to each other) to $\partial\mathcal{M}_R$ and $\partial\mathcal{M}_L$
- path integrals on $\mathcal{M}_R$ and $\mathcal{M}_L$ give vectors ψ and χ in $\mathcal{H}_R$ and $\mathcal{H}_L$
- before gluing back the manifolds, act on the boundary $\partial\mathcal{M}_R$ with a diffeomorphism K, $\psi \to K\psi$

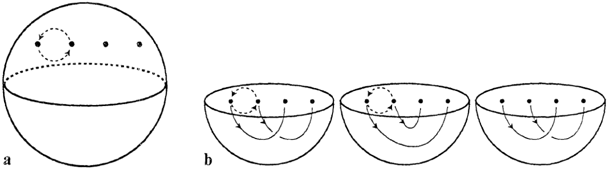
Step III: Surgery – glue the manifold

- connected sum: $\mathcal{M} = \mathcal{M}_L \# \mathcal{M}_R / X_1 / X_2$
- natural pairing: partition function $Z(\mathcal{L}) = (\chi, K\psi)$
- in a 2d vector space, any 3 vectors ψ, ψ_1, ψ_2 in $\mathcal{H}_R$ obey a linear dependence relation,

$$\alpha\psi + \beta\psi_1 + \gamma\psi_2 = 0$$
$$\alpha Z(\mathcal{L}) + \beta Z(\mathcal{L}_1) + \gamma Z(\mathcal{L}_2) = 0$$



Step IV: Find the coefficients α, β & γ



- **half-monodromy** B: $\psi_1 = B\psi, \quad \psi_2 = B^2\psi$
  two points undergo a half-twist/diffeomorphism about one another
- matrix B acts in a 2-dimensional space & obeys a characteristic equation

$$B^2 - \mathrm{tr}(B)B + \det(B) = 0$$

- $\det(B)\psi - \mathrm{tr}(B)\psi_1 + \psi_2 = 0$ [cf. $\alpha Z(\mathcal{L}) + \beta Z(\mathcal{L}_1) + \gamma Z(\mathcal{L}_2) = 0$]
- $\alpha = \det(B), \quad \beta = -\mathrm{tr}(B), \quad \gamma = 1$

[Moore & Seiberg, Phy. Lett. B, 1988]

- $\alpha = \det(B), \quad \beta = -\operatorname{tr}(B), \quad \gamma = 1$
- need only the eigenvalues of B: $\lambda_i = \pm \exp(i\pi(2h_{\mathcal{R}} - h_{E_i}))$
  $h_{\mathcal{R}}$: **conformal weight** of the WZW primary field corresponding to $\mathcal{R}$

$$h_{\mathcal{R}} = \frac{N^2 - 1}{2N(k+N)} \quad N = 2 \text{ for } \mathrm{SU}(2)$$

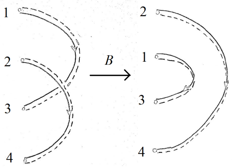
  E_i: irreps of SU(2), $\mathcal{R} \otimes \mathcal{R} = E_1 \oplus E_2$
  h_{E_i}: weights of primary fields corresponding to $\mathcal{R} \otimes \mathcal{R}$

$$h_{E_1} = \frac{N^2 + N - 2}{N(k+N)}$$
$$h_{E_2} = \frac{N^2 - N - 2}{N(k+N)} \quad N = 2 \text{ for } \mathrm{SU}(2)$$

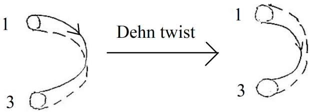
- $\lambda_1 = \exp\left(\frac{-i\pi}{2(k+2)}\right), \quad \lambda_2 = -\exp\left(\frac{3i\pi}{2(k+2)}\right)$
- $\alpha = -\exp\left(\frac{i\pi}{k+2}\right), \quad \beta = -\exp\left(\frac{-i\pi}{2(k+2)}\right) + \exp\left(\frac{3i\pi}{2(k+2)}\right), \quad \gamma = 1$

Step V: Fix the change of framing

- A twist is induced in the framing after the operation of half-monodromy B



- **Dehn twist** reverts the framed link to its original framing



- multiply with a factor $\exp(-2\pi i h_{\mathcal{R}}) = \exp\left(\frac{-3\pi i}{k+2}\right)$
- $\alpha = -\exp\left(\frac{\pi i}{k+2}\right), \quad \beta = -\exp\left(\frac{-2\pi i}{k+2}\right) + 1, \quad \gamma = \exp\left(\frac{-3\pi i}{k+2}\right)$

- mult. with $\exp(\frac{\pi i}{k+2})$, subs. $t = \exp(\frac{2\pi i}{k+2})$, replace $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ by $\mathcal{L}_+, \mathcal{L}_0, \mathcal{L}_-$

$$-tZ(\mathcal{L}_+) + (t^{1/2} - t^{-1/2})Z(\mathcal{L}_0) + t^{-1}Z(\mathcal{L}_-) = 0$$

- identification $V(\mathcal{L}) = \frac{Z(\mathcal{L})}{Z(\text{unknotted Wilson loop})}$

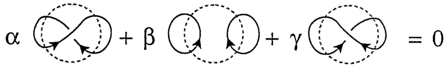
$$-tV(\mathcal{L}_+) + (t^{1/2} - t^{-1/2})V(\mathcal{L}_0) + t^{-1}V(\mathcal{L}_-) = 0$$

- correlation function $W_{\square\ldots\square}(\mathcal{L}) = Z(\mathrm{S}^3; \mathcal{L})/Z(\mathrm{S}^3)$

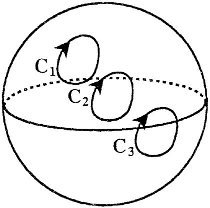
$$W_{\square\ldots\square}(\mathcal{L}) = t^{2\,lk(\mathcal{L})}\left(\frac{t - t^{-1}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right)V^{\mathcal{L}}(t)$$

[Mariño, hep-th/0406005]

Example: Unknot



$$\alpha Z(S^3; C) + \beta Z(S^3; C^2) + \gamma Z(S^3; C) = 0$$



$$\frac{Z(S^3;\ C_1 \ldots C_s)}{Z(S^3)} = \prod_{k=1}^{s} \frac{Z(S^3;\ C_k)}{Z(S^3)}$$

$$\frac{Z(S^3;\ C)}{Z(S^3)} = -\frac{\alpha + \gamma}{\beta} = t^{1/2} + t^{-1/2}, \quad t = \exp\left(\frac{2\pi i}{k+2}\right)$$

$$(\text{cf. } W_{\square}(C) = t^{2\,lk(C)} \left(\frac{t - t^{-1}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right) V^C(t), \quad lk(C) = 0, V^C(t) = 1)$$

CS theory, $G = SU(2)$, fund. reps $\longleftrightarrow$ Jones polynomial

$$W_{\Box\cdots\Box}(\mathcal{L}) = t^{2\,lk(\mathcal{L})}\left(\frac{t - t^{-1}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}\right)V^{\mathcal{L}}(t),$$

$$t = \exp\left(\frac{2\pi i}{k+2}\right)$$

CS theory, $G = SU(N)$, fund. reps $\longleftrightarrow$ HOMFLY(PT) polynomial

$$W_{\Box\cdots\Box}(\mathcal{L}) = \lambda^{lk(\mathcal{L})}\left(\frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\right)P^{\mathcal{L}}(q,\lambda),$$

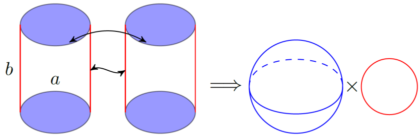
$$q = \exp\left(\frac{2\pi i}{k+N}\right), \quad \lambda = q^N$$

CS theory, $G = SO(N)$, fund. reps $\longleftrightarrow$ Kauffman polynomial

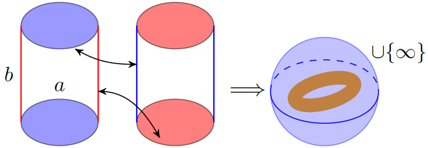
CS theory, $G = SU(2)$, higher dim. reps $\longleftrightarrow$ Akutsu-Wadati polynomials

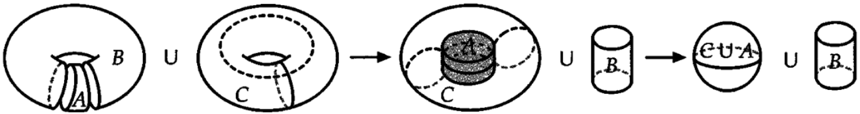
Partition function of S^3

Gluing two solid tori ($D \times S^1$), a-cycle to a-cycle and b-cycle to b-cycle.



Gluing two solid tori after an S-transformation. We consider the solid blue ball $\cup\{\infty\}$ as S^3, in which there is a brown solid torus embedded.





S-matrix elements in the basis of integrable representations of the affine Lie algebra associated to $G = SU(2)$ at level k:

$$S_{ab} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(a+1)(b+1)\pi}{k+2}\right)$$

$$Z(\mathrm{S}^2 \times \mathrm{S}^1) = \langle 0|0\rangle = 1$$

$$Z(\mathrm{S}^3) = \langle 0|S|0\rangle = S_{00} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right)$$

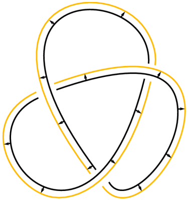
$$W_{\square}(\bigcirc) = \frac{Z(\mathrm{S}^3; \bigcirc)}{Z(\mathrm{S}^3)} = \frac{\langle 0|S|\mathcal{R}\rangle}{\langle 0|S|0\rangle} = \frac{S_{0\mathcal{R}}}{S_{00}} = \frac{S_{01}}{S_{00}}$$

$$= t^{1/2} + t^{-1/2}, \quad t = \exp\left(\frac{2\pi i}{k+2}\right)$$

Framing Dependence of Links and Manifolds

Links and manifolds are assumed to be framed

- Link: fix a trivialisation of its normal bundle in link $\mathcal{L}$ (standard/blackboard framing of $\mathcal{L}$)
- Manifold: fix a trivialisation of the tangent bundle (parallelisable manifold, admitting a global field of frames/linearly independent vector fields at each point)



Thanks!