Calabi-Yau Compactification:
A Primer on Calabi-Yau Manifolds

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Abstract: This article consists of an introduction to Calabi-Yau manifolds that are manifest in Type I and heterotic superstring theories. The unbroken $\mathcal{N} = 1$ supersymmetry conditions are discussed to motivate the construction of Calabi-Yau three-folds. Properties of Kähler and Calabi-Yau manifolds are also described, along with some concrete examples.

Contents

1 Introduction

Since the 1970s, superstring theories have been good candidates for mathematically consistent theories of quantum gravity. The theories are formulated in ten dimensions and reduce to ten-dimensional supergravity theories at low energies. We would like to make sense of this as realistic four-dimensional physics. In other words, we will compactify the ten-dimensional theories on a six-dimensional space $\mathcal{K}$, which can be determined uniquely via some extremely restrictive requirements:

1. The geometry should be of the form $\mathcal{M}_4 \times \mathcal{K}$, with $\mathcal{M}_4$ being a maximally symmetric spacetime.
2. There should be an unbroken $\mathcal{N} = 1$ supersymmetry in four dimensions.
3. The gauge group and fermion spectrum should be realistic.

The Green-Schwarz mechanism [1], an anomaly cancellation procedure in $d = 10$ Type I and heterotic $SO(32)$ and $E_8 \times E_8$ theories, satisfies precisely these requirements.

It is necessary that the vacuum state be of the form $\mathcal{M}_4 \times \mathcal{K}$ which maintains four-dimensional Poincaré invariance. As we will see, $\mathcal{M}_4$ is our favourite four-dimensional Minkowski space and $\mathcal{K}$ is some compact six-dimensional manifold; all dimensions are on the same logical footing following Kaluza-Klein interpretation [2, 3]. Of course, physical fluctuations will not necessarily respect the product form of the vacuum configuration, but understanding the ground state is the key to understanding the low-energy excitations.

Superstrings can move only in geometries that satisfy the equations of motion of ten-dimensional supergravity. If we further require the absence of matter, this means that the metric needs to be *Ricci flat* so that it satisfies the vacuum Einstein's equations. We shall look for the Ricci flat geometries in a certain class of complex manifolds, known as the Kähler manifold. In the mathematics literature, Ricci flat Kähler manifolds have been dubbed Calabi-Yau spaces, whose existence was conjectured by Calabi [4] and proved by Yau [5].

Section 2 will be devoted to the motivations behind the construction of Calabi-Yau manifolds from the field theory viewpoint. These are in fact the unbroken $\mathcal{N}=1$ supersymmetry conditions. These conditions will in turn require that for perturbatively accessible configurations, $\mathcal{K}$ has $SU(3)$ holonomy and that the four-dimensional cosmological constant vanish. The properties of Kähler and Calabi-Yau manifolds $\mathcal{K}$ will be discussed in Sections 3 and 4.

We would like to mention two important results which will not be discussed in this paper. First, on Calabi-Yau manifolds, the theory leads naturally to a $d=4$ theory with E_6 gauge group and chiral fermion generations. Second, similar conclusions can be reached from a stringy point of view. For references, readers may consult [6] and Chapters 14-16 in [7], on which the contents of this paper are based.

2 Unbroken $\mathcal{N}=1$ Supersymmetry

The unbroken $\mathcal{N}=1$ supersymmetry in four dimensions interests us for several reasons.

Motivations

1. One striking motivation is to solve the gauge-hierarchy problem, which questions why the mass scale of weak-interaction symmetry breaking is so tiny compared to more fundamental scales such as the Planck mass. In particular, to explain the existence of massless charged spin zero fields ($SU(2)\times U(1)$ Higgs doublet), one needs to postulate the existence of an unbroken supersymmetry of the low-energy world. Under this assumption, massless charged scalars can naturally arise as supersymmetric partners of massless charged fermions.

2. A state of unbroken supersymmetry in four dimensions always obeys the equations of motion of the higher dimensional string and supergravity theories. This is most obvious in global supersymmetry, where the Hamiltonian is positive semi-definite and vanishes when and only when supersymmetry is unbroken.

3. The hypothesis of unbroken $\mathcal{N}=1$ supersymmetry in four dimensions is very restrictive, but not too restrictive, for phenomenology. In four-dimensional supersymmetric theories with $\mathcal{N}>2$ supersymmetry, the massless fermions always transform in a real representation of the gauge group, in stark contrast to what is observed in nature.

We will concentrate on the $SO(32)$ and $E_8\times E_8$ theories because they have elementary gauge fields and can generate chiral fermions in four dimensions. We note that many considerations that we will develop could be carried out for the type II theories as well.

Setup

1. We shall use the following conventions.

   (a) Upper case Latin indices refer to the entire ten-dimensional spacetime, lower case Latin indices refer to $\mathcal{K}$, and Greek indices refer to $\mathcal{M}_4$.

   (b) Components of tensors on $\mathcal{K}$ are denoted $m, n, \cdots$ in a real basis and i, j, k in a complex basis.

   (c) The spinor representation of $SO(10)$ or $SO(1,9)$ is of dimension $2^5 = 32$. The Dirac matrices obey $\left\{\Gamma^M, \Gamma^N\right\} = 2g^{MN}$ with signature $(-,+,+,\cdots,+)$. We employ a Majorana representation with Γ^M being real, hermitian 32×32 matrices, apart from Γ^0 which is real and antihermitian. The Γ^M may also be represented as tensor products of the matrices γ^μ appropriate to Minkowski space, with the matrices γ^m appropriate to the internal space:

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1}, \quad \Gamma^m = \gamma_5 \otimes \gamma^m, \tag{2.1}$$

   with

$$\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}. \tag{2.2}$$

   We also refer to the matrix

$$\gamma = \frac{i}{6!}\sqrt{g}\epsilon_{mnpqrs}\gamma^{mnpqrs}, \tag{2.3}$$

   which determines the chirality in the internal space. Thus γ^μ's are real and hermitian, apart from γ^0 which is antihermitian, and γ^m's are imaginary and hermitian as are γ_5 and γ.

2. In the low-energy effective field theories in ten dimensions,

   (a) the $d = 10$, $\mathcal{N} = 1$ supergravity multiplet contains a metric g_{MN}, spin-$\frac{3}{2}$ gravitino ψ_M, two-form B_{MN}, spin-$\frac{1}{2}$ dilatino λ, and scalar field ϕ.

   (b) the super Yang-Mills multiplet contains the Yang-Mills field strength F^a_{MN} and spin-$\frac{1}{2}$ gluino χ^a.

3. We will require the background field configuration to be maximally symmetric on $\mathcal{M}_4$, which requires the background values of the Fermi fields to vanish.

4. An unbroken supersymmetry is simply a conserved supercharge Q that annihilates the vacuum state $|\Omega\rangle$, which is equivalent to saying that for all operators U, $\langle\Omega|\{Q,U\}|\Omega\rangle = 0$. This will certainly be so for a bosonic operator U, since then $\{Q,U\}$ is fermionic. On the other hand, when U is a fermionic operator, $\{Q,U\}$ is simply $\delta_\varepsilon U$, the variation of U under the unbroken supersymmetry transformation generated by ε. So finding an unbroken supersymmetry at tree level means finding a supersymmetry transformation such that $\delta_\varepsilon U = 0$ for every fermionic field U.

In the field theory limit of string theory, under the assumption of maximal symmetry, the relevant supersymmetry variations for the Fermi fields are

$$\delta\psi_\mu = \nabla_\mu\varepsilon + \frac{\sqrt{2}}{32}e^{2\phi}\left(\gamma_\mu\gamma_5\otimes H\right)\varepsilon, \tag{2.4a}$$

$$\delta\psi_m = \nabla_m\varepsilon + \frac{\sqrt{2}}{32}e^{2\phi}\left(\gamma_m H - 12H_m\right)\varepsilon, \tag{2.4b}$$

$$\delta\lambda = \sqrt{2}\left(\gamma^m\nabla_m\phi\right)\varepsilon + \frac{1}{8}e^{2\phi}H\varepsilon, \tag{2.4c}$$

$$\delta\chi^a = -\frac{1}{4}e^{\phi}F^a_{mn}\gamma^{mn}\varepsilon. \tag{2.4d}$$

Here, ϕ is the dilaton field and H_{pqr} is the gauge-invariant field strength of the two-from B_{pq} the antisymmetric tensor field strength (often called the *torsion* in the literature), with the contractions

$$H = H_{pqr}\gamma^{pqr}, \quad H_m = H_{mqr}\gamma^{qr}. \tag{2.5}$$

The four transformation laws were derived for the ten-dimensional coupled supergravity-super Yang-Mills field theory formulated by Chapline and Manton [8]. For the field strength H to be Lorentz invariant under the Lorentz transformation $\delta_L B = \mathrm{tr}\,(\Theta\ d\hat{\omega})$ ($\hat{\omega}$ is the spin connection; Θ is a matrix of infinitesimal parameters) in Green-Schwarz mechanism, it should take the form

$$H = dB - \omega_{3Y} + \omega_{3L}, \tag{2.6}$$

where ω_{3Y} is the Yang-Mills Chern-Simon three-form,

$$\omega_{3Y} = \frac{1}{30}\,\mathrm{Tr}\left(A\wedge F - \frac{1}{3}A\wedge A\wedge A\right) = \frac{1}{30}\,\mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right), \tag{2.7}$$

and ω_{3L} is the Lorentz Chern-Simon three-form,

$$\omega_{3L} = \frac{1}{30}\,\mathrm{Tr}\left(\hat{\omega}d\hat{\omega} + \frac{2}{3}\hat{\omega}^3\right). \tag{2.8}$$

Here Tr denotes a trace in the adjoint representation of either $SO(32)$ or $E_8\times E_8$. Acting with an exterior derivative on (2.6) gives the Bianchi identity for H,

$$\begin{aligned} dH &= \mathrm{tr}\,R\wedge R - \mathrm{tr}\,F\wedge F \\ &= \mathrm{tr}\,R\wedge R - \frac{1}{30}\,\mathrm{Tr}\,F\wedge F, \end{aligned} \tag{2.9}$$

where we have a trace in the vector representation of $SO(1,9)$, $\mathrm{tr}\,R\wedge R = d\omega_{3L}$, and a trace in the vector representation of $SO(32)$, $\mathrm{tr}\,F\wedge F = d\omega_{3Y}$.

Analysis

For some nonzero choice of ε, the vanishing of the supersymmetry variations $\delta\psi_\mu, \delta\psi_m, \delta\lambda$ and $\delta\chi^a$ leads to some important restrictions on the Yang-Mills field strength F and the geometry of both $\mathcal{K}$ and $\mathcal{M}_4$.

1. **Vanishing of $\delta\psi_\mu$**

   Taking the commutator of (2.4a), we find the integrability condition

$$[\nabla_\mu, \nabla_\nu]\,\varepsilon - \frac{1}{(16)^2} e^{4\phi} \left(\gamma_{\mu\nu} \otimes H^2\right) \varepsilon = 0. \tag{2.10}$$

   Assuming that $\mathcal{M}_4$ is a maximally symmetric space, we have the curvature tensor

$$R_{\mu\nu\rho\sigma} = \kappa \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right), \tag{2.11}$$

   with $\kappa = \frac{R^{(4)}}{12}$ a parameter that is positive for de Sitter space, negative for anti-de Sitter space, and zero for Minkowski space ($R^{(4)}$ being the four-dimensional Ricci scalar). Since

$$[\nabla_\mu, \nabla_\nu]\,\varepsilon = \frac{1}{4} R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} \varepsilon = \frac{1}{2} \kappa \gamma_{\mu\nu} \varepsilon, \tag{2.12}$$

   we must have

$$\left[\gamma_{\mu\nu} \otimes \left(\frac{1}{2}\kappa \mathbf{1} - \frac{1}{(16)^2} e^{4\phi} H^2\right)\right] \varepsilon = 0. \tag{2.13}$$

   For $\mu \neq \nu$, $\gamma_{\mu\nu}$ is invertible so we find that ε is an eigenspinor of H^2:

$$H^2 \varepsilon = \frac{1}{2} \kappa (16)^2 e^{-4\phi} \varepsilon. \tag{2.14}$$

   Note that H is an antihermitian matrix and thus $\kappa \leq 0$.

2. **Vanishing of $\delta\lambda$**

   This condition implies that

$$\left[\sqrt{2}\left(\gamma^m \nabla_m \phi\right) + \frac{1}{8} e^{2\phi} H\right]^2 \varepsilon = 0, \tag{2.15}$$

   which together with (2.14) gives

$$\frac{1}{8}\sqrt{2} e^{2\phi} \left\{\gamma^m \nabla_m \phi, H\right\} \varepsilon = -2 \left(\nabla_m \phi \nabla^m \phi + \kappa\right) \varepsilon, \tag{2.16}$$

   hence ε is also an eigenspinor of $\{\gamma^m \nabla_m \phi, H\}$. Since H is antihermitian, we can have only imaginary eigenvalues. It follows that the right-hand side of (2.16) must vanish:

$$\nabla_m \phi \nabla^m \phi = -\kappa, \tag{2.17}$$

   and so $\nabla_m \phi \nabla^m \phi$ must be a constant. However since ϕ depends only on the coordinates of the internal manifold $\mathcal{K}$, which is compact, ϕ must have a maximum somewhere. Thus κ must vanish and ϕ must be a constant:

$$\nabla_m \phi = 0, \quad \kappa = 0. \tag{2.18}$$

   Therefore $\mathcal{M}_4$ is in fact flat! From (2.14) we also have

$$H\varepsilon = 0. \tag{2.19}$$

3. **Vanishing of $\delta\psi_m$**

The variation (2.4b) now reduces to

$$\tilde{\nabla}_m \varepsilon \equiv \nabla_m \varepsilon - \beta H_m \varepsilon = 0, \quad \beta = \frac{3}{8}\sqrt{2} e^{2\phi}, \tag{2.20}$$

where $\tilde{\nabla}_m$ is defined to be the covariant derivative with torsion $-4\beta H_{pqr}$. The existence of a spinor that satisfies (2.19) and (2.20) has important geometrical and topological consequences which are associated with the fact that $\mathcal{K}$ is a complex manifold. To show this, we will now derive a number of useful identities.

In virtue of (2.19) and (2.20), we have

$$\gamma^m \nabla_m \varepsilon = 0 \tag{2.21}$$

and by operating with $\gamma^n \nabla_n$,

$$\left(\nabla^m \nabla_m - \frac{1}{4} R \right) \varepsilon = 0. \tag{2.22}$$

However if we take the divergence of (2.20) directly we find

$$\left(\nabla^m \nabla_m - \beta H^m{}_{;m} - \beta^2 H^m H_m \right) \varepsilon = 0. \tag{2.23}$$

Comparing with (2.22) and using the identity

$$H_m H^m = \frac{1}{9} H^2 - \frac{4}{3} H_{pqr} H^{pqr}, \tag{2.24}$$

we have

$$\left(\beta H^m{}_{;m} - \frac{1}{4} R - \frac{4}{3} \beta^2 H_{pqr} H^{pqr} \right) \varepsilon = 0. \tag{2.25}$$

However $H^m{}_{;m}$ is an antihermitian matrix and so we can have only imaginary eigenvalues. It follows that

$$\begin{aligned} H^m{}_{;m} \varepsilon &= 0, \\ R &= -\frac{16}{3} \beta^2 H_{pqr} H^{pqr}. \end{aligned} \tag{2.26}$$

Considering again (2.20), we take its commutator with $\tilde{\nabla}_n$, which yields the relation

$$\tilde{R}_{mnpq} \gamma^{pq} \varepsilon = 0, \tag{2.27}$$

where $\tilde{R}_{mnpq}$ is the Riemann tensor constructed from the connection with torsion.

4. **Vanishing of $\delta\chi^a$**

This condition implies that the field strength F must satisfy

$$F_{mn}{}^a \gamma^{mn} \varepsilon = 0. \tag{2.28}$$

Remark. *Under the decomposition* $SO(9,1) \to SO(3,1) \times SO(6) \simeq SO(3,1) \times SU(4)$*, the* **16** *decomposes as*

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}). \tag{2.29}$$

This is the Majorana-Weyl **16** *supersymmetry parameter (or a-number spinor)* ε*, which is real and subject to the Weyl constraint:* $(\gamma_5 \otimes \gamma)\,\varepsilon = -\varepsilon$*. Such a spinor can be written as*

$$\varepsilon(y) \to \varepsilon_{\alpha\beta}(y) + \varepsilon^*_{\alpha\beta}(y), \quad y = (y^1, \cdots, y^6) \in \mathcal{K}, \tag{2.30}$$

where the indices on $\varepsilon_{\alpha\beta}$ *transform respectively as* $(\mathbf{2}, \mathbf{4})$*. If there is any unbroken supersymmetry, then by* $SO(3,1)$ *rotations we can generate further supersymmetries and so reach the form*

$$\varepsilon_{\alpha\beta}(y) = \xi_\alpha \eta_\beta(y), \tag{2.31}$$

with the ξ_α *a set of four real, constant, linearly independent, a-number spinors (transform as* $\mathbf{2} \oplus \bar{\mathbf{2}}$*) and the* η_β *a set of four real c-number spinors (transform as* $\mathbf{4} \oplus \bar{\mathbf{4}}$*). The reality of* ξ *and* η *prevents them from being eigenspinors of* $\gamma_5 \otimes \gamma$*. Hence if there exists a spinor* ε *satisfying our requirements, then there must exist on the internal space at least two real c-number spinors* η *satisfying the same requirements. From now on, we will focus on the internal space spinor* η*.*

Vanishing H

We may simplify things by considering the case $H_{pqr} = 0$. We note that the right-hand side of (2.9) scales differently from the left-hand side under dilations $e^{-\phi} r \to \lambda(e^{-\phi} r)$, where r is the radius of curvature of $\mathcal{K}$. Solutions with non-zero H_{pqr} will be of some fixed value of $e^{-\phi} r$, generically of order the ten-dimensional Planck length ℓ_p and higher derivative corrections to the supersymmetry transformation laws (2.4) would be important. So such solutions would not in general be reliable. By contrast, for zero H_{pqr}, the value of $e^{-\phi} r$ is not fixed. When this scale is large compared to ℓ_p, field theory considerations are reliable.

In this case, from (2.20) and (2.31), one has a covariantly constant spinor:

$$\nabla_m \eta = 0. \tag{2.32}$$

This condition is very restrictive where it implies (2.27):

$$R_{mnpq} \gamma^{pq} \eta = 0, \tag{2.33}$$

which upon contracting with γ^n gives

$$\gamma^n \gamma^{pq} R_{mnpq} \eta = 0. \tag{2.34}$$

Using the gamma matrix identity $\gamma^n \gamma^{pq} = \gamma^{npq} + g^{np}\gamma^q - g^{nq}\gamma^p$, and the curvature identity $R_{mnpq} + R_{mqnp} + R_{mpqn} = 0$, (2.34) implies that

$$\gamma^p R_{mp} \eta = 0. \tag{2.35}$$

Hence, the internal manifold $\mathcal{K}$ is shown to be **Ricci-flat**:

$$R_{mn} = 0. \tag{2.36}$$

In terms of such η we may construct an **almost complex structure**:

$$J_m{}^n = -i\eta^\dagger \gamma_m{}^n \gamma \eta, \tag{2.37}$$

which, by the Fierz rearrangement, satisfies

$$J_m{}^n J_n{}^u = -\delta_m{}^u \quad (J^2 = -\mathbf{1}). \tag{2.38}$$

The metric is **hermitian**[1] with respect to the almost complex structure J:

$$J_m{}^p J_n{}^q g_{pq} = g_{mn}, \tag{2.39}$$

An integrable[2] almost complex structure is called a **complex structure**. A manifold $\mathcal{K}$ endowed with a complex structure J is then called a **complex manifold**. A necessary and sufficient condition for the almost complex structure J to be integrable, so that the manifold can be a complex manifold, is the vanishing of a curvature-like tensor associated with the almost complex structure, known as the **Nijenhuis tensor**:

$$N_{mn}{}^p = J_m{}^q J_{[n}{}^p{}_{;q]} - J_n{}^q J_{[m}{}^p{}_{;q]}. \tag{2.40}$$

Indeed, with (2.20) and (2.37), we have

$$\nabla_p J_{mn} = -8\beta H_{sp[m} J_{n]}{}^s, \tag{2.41}$$

from which it follows that $N_{mn}{}^p$ is proportional to $\eta^\dagger \{H, \gamma_{mn}{}^p\} \eta$ and hence vanishes. Thus $\mathcal{K}$ is a complex manifold with a hermitian metric. The condition (2.32) also implies that J is covariantly constant. This leads to construction of the so-called **Kähler manifold** $\mathcal{K}$ (see next section).

Furthermore, the vanishing of H_{pqr} requires that the difference of the Yang-Mills and Lorentz Chern-Simons forms should be the exterior derivative of a two-form. This has as its integrability condition that

$$\frac{1}{30} \operatorname{Tr} F \wedge F = \operatorname{tr} R \wedge R. \tag{2.42}$$

This is the additional condition that the Yang-Mills field must also satisfy in addition to (2.28) with our assumption that $H = 0$. These two conditions are very restrictive that they lead to a four-dimensional phenomenological (realistic) model.

[1] In a given coordinate patch of a complex manifold the most general metric takes the form $g = g_{ij} dz^i \otimes dz^j + g_{\bar{i}\bar{j}} d\bar{z}^{\bar{i}} \otimes d\bar{z}^{\bar{j}} + g_{\bar{i}j} d\bar{z}^{\bar{i}} \otimes dz^j + g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}$. A **hermitian metric** on a complex manifold is one for which $g_{ij} = 0 = g_{\bar{i}\bar{j}}$, i.e. $g = g_{\bar{i}j} d\bar{z}^{\bar{i}} \otimes dz^j + g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}$. It provides us with a pairing between holomorphic and antiholomorphic indices. When we lower or raise holomorphic indices they become antiholomorphic and vice versa.

[2] At any point y in $\mathcal{K}$, there is a suitable basis of complex coordinates z^i and their complex conjugates $\bar{z}^{\bar{j}}$ $(i, \bar{j} = 1, 2, 3)$, in which the almost complex structure J takes the form $J^i{}_j = i\delta^i{}_j, J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}{}_{\bar{j}}$ with other components zero. We will call this the canonical form of J. If J can be expressed in the canonical form not just at one point $p \in \mathcal{K}$ but in a whole open set containing p, then the complex coordinates may be called the local holomorphic coordinates. If local holomorphic coordinates exist (in a neighborhood of each point p), then J is said to be integrable.

$SU(3)$ Holonomy

We showed that $\mathcal{K}$ *is Ricci-flat and Kähler when the field strength H vanishes*. However, a general compact manifold does not admit a Ricci-flat Kähler metric because of certain topological obstructions. The nature of these obstructions can be most easily understood in terms of the holonomy group. We will therefore reformulate our conditions on $\mathcal{K}$ in the language of holonomy.

On any curved manifold, tangent vectors parallel-transported around closed loops undergo rotations from their original orientations. The group of such rotations is called the tangent-space group or the **holonomy group**, H. For a six-dimensional manifold $\mathcal{K}$, the spin connection $\hat{\omega}$ is *a priori* an $SO(6)$ gauge field. Upon parallel transport around a closed loop L, a physical field ψ is transformed into $U\psi$, where $U = P \exp \int_L \hat{\omega} \cdot dx$ is the path-ordered exponential of $\hat{\omega}$ around the loop L. These $SO(6)$ matrices U always form the holonomy group, H of the manifold $\mathcal{K}$.

We would expect that the rotations would fill out the whole of $SO(6)$ for a general six-dimensional manifold. However, this is not so for a covariantly constant spinor field η, which always returns to its original value upon parallel transport around a loop, viz. $U\eta = \eta$. We note that the Lie algebra of $SO(6)$ is isomorphic to that of $SU(4)$. The positive and negative chirality spinor of $SO(6)$ are the fundamental $\mathbf{4}$ and $\overline{\mathbf{4}}$ of $SU(4)$. Thus a covariantly constant spinor may be thought of as a preferred orientation in this $SU(4)$ which is not changed by parallel transport.

We can always put η in the form $\eta = (0, 0, 0, \eta_0)^T$, and the unbroken $SU(3)$ group is simply the subgroup[3] (little group) of $SU(4)$ that acts on the first three components. The existence of a covariantly constant η thus means that the spin connection $\hat{\omega}$ only takes values in the little group of η, i.e. it is an $SU(3)$ gauge field. In other words, this means that the manifold has $SU(3)$ *holonomy*. A manifold whose holonomy group is $SU(3)$ has precisely two covariantly constant spinors η and $i\gamma\eta$ since under $SU(3)$ the $\mathbf{4} \oplus \overline{\mathbf{4}}$ representation of $SU(4)$ decomposes as $\mathbf{4} \oplus \overline{\mathbf{4}} = \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{1} \oplus \mathbf{1}$.

3 Kähler Manifolds

Let us consider a manifold whose holonomy is $U(3)$ (the more restrictive case of $SU(3)$ will be discussed in the next section). Under $U(3)$, the vector representation of $SO(6)$ decomposes as $\mathbf{6} \cong \mathbf{3} \oplus \overline{\mathbf{3}}$. In particular, in a system of local holomorphic coordinates z^i, and their complex conjugates $\bar{z}^{\bar{j}}$ $(i, \bar{j} = 1, 2, 3)$, a vector V^m can be split into holomorphic components V^i and antiholomorphic components $V^{\bar{j}}$.

The distinction between holomorphic and antiholomorphic indices is invariant under holomorphic changes of coordinates and does not depend on the choice of a particular holomorphic

[3]This is analogous to the fact that in gauge theories, the group $SU(4)$ can be broken down to $SU(3)$ by the expectation value of a Higgs $\mathbf{4}$.

coordinate system. Hence, it is very natural to decompose tensor fields into pieces with definite numbers of holomorphic and antiholomorphic indices. In particular, the metric tensor g_{mn} at a point $p \in \mathcal{K}$ is invariant under the action of $U(3)$ on the tangent space at p. The metric g obeys $g_{ij} = g_{\bar{i}\bar{j}} = 0$ since we can only make a $U(3)$ singlet[4] from $\mathbf{3} \otimes \bar{\mathbf{3}}$. Also, being symmetric, g obeys $g_{i\bar{j}} = g_{\bar{j}i}$. This implies that $\mathcal{K}$ *is a complex manifold with a hermitian metric.*

Kähler Metric

In fact, $U(3)$ holonomy further implies that $\mathcal{K}$ is a **Kähler manifold**, which is a complex manifold that admits a special form of hermitian metric, known as the Kähler metric. A general local expression for the **Kähler metric** is

$$g_{i\bar{j}} = g_{\bar{j}i} = \frac{\partial^2 \Phi(z, \bar{z})}{\partial z^i \partial \bar{z}^{\bar{j}}}, \tag{3.1}$$

where $\Phi(z, \bar{z})$ is a (0,0) form or a scalar function, known as the **Kähler potential**. We shall prove (more generally, for a manifold of $U(N)$ holonomy) why we can write the Kähler metric in the form (3.1).

Claim. *Any metric can be put locally in the form of (3.1) (with $g_{ij} = g_{\bar{i}\bar{j}} = 0$) if and only if it has $U(N)$ holonomy.*

Proof. On a manifold $\mathcal{K}$ of $U(N)$ holonomy, we can define a tensor field

$$J^m{}_n(y) = g^{mp}(y)\eta^\dagger(y)\gamma_{pn}\eta(y). \tag{3.2}$$

(This is just a redefinition of the almost complex structure (2.37) in terms of the metric.) For each $y \in \mathcal{K}$, $J^m{}_n$ can be viewed as a matrix acting on tangent vectors, the action being $V^m \to J^m{}_n V^n$ for any tangent vector V^m. $J^m{}_n$ is in fact a matrix that assigns the value i or $-i$ to states in the vectors $\mathbf{N}$ or $\overline{\mathbf{N}}$ of $U(N)$. A $U(N)$ holonomy thus means that $J^m{}_n$ is covariantly constant, and the Nijenhuis tensor

$$N^p{}_{mn} = J^q{}_m(\partial_q J^p{}_n - \partial_n J^p{}_q) - J^q{}_n(\partial_q J^p{}_m - \partial_m J^p{}_q) \tag{3.3}$$

vanishes. Now define a $(1,1)$ form, called the **Kähler form**, $k_{mn} = -g_{mp}J^p{}_n$, and using the explicit form of J in local holomorphic coordinates, $J^i{}_j = i\delta^i{}_j, J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}{}_{\bar{j}}$, we find that

$$k_{ij} = k_{\bar{i}\bar{j}} = 0, \quad k_{i\bar{j}} = ig_{i\bar{j}} = -k_{\bar{j}i}. \tag{3.4}$$

By the symmetry of g, we can write the Kähler form[5] as $k = ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$. Since g and J are covariantly constant, k is covariantly constant (or closed $dk = 0$); it obeys $\partial k = \bar{\partial} k = 0$. It

[4] In general, it is impossible to make a $U(N)$ singlet from $\mathbf{N} \otimes \mathbf{N}$ or $\overline{\mathbf{N}} \otimes \overline{\mathbf{N}}$ for $\mathbf{N} > \mathbf{2}$.

[5] If the Kähler form is closed, $dk = 0$, then we have a Kähler manifold. Here, the exterior derivative d can be written as $d = \partial + \bar{\partial}$, where ∂ and $\bar{\partial}$ are respectively the holomorphic and anti-holomorphic exterior derivatives, often called the Dolbeault operators.

follows then that k can be expressed locally in terms of a scalar function $\Phi(z, \bar{z})$ as

$$k = i\partial\bar{\partial}\Phi(z, \bar{z}). \tag{3.5}$$

Bearing in mind the relation $k_{i\bar{j}} = ig_{i\bar{j}}$, we see from (3.5) that the metric $g_{i\bar{j}}$ is of the form (3.1).

To prove the converse, it is useful to first work out the form of the affine connection for Kähler manifolds. One easily finds that the only nonzero components of the affine connection $\Gamma^p{}_{mn} = \frac{1}{2}g^{pq}(\partial_m g_{qn} + \partial_n g_{qm} - \partial_q g_{mn})$ are

$$\Gamma^i{}_{jk} = g^{i\bar{l}}\partial_j g_{k\bar{l}}, \quad \Gamma^{\bar{i}}{}_{\bar{j}\bar{k}} = g^{\bar{i}l}\partial_{\bar{j}} g_{\bar{k}l}. \tag{3.6}$$

It follows from this form of the affine connection that the standard complex structure J (with nonzero components $J^i{}_j = i\delta^i{}_j, J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}{}_{\bar{j}}$) is covariantly constant. Since the subgroup of $SO(2N)$ under which J is invariant is $U(N)$, covariant constancy of J means that the holonomy group of a Kähler manifold is (at most) $U(N)$, completing the demonstration that *metrics of $U(N)$ holonomy are precisely Kähler metrics.* □

Remark. *The Kähler metric is not unique; it is always possible to modify the Kähler potential Φ. If f is an arbitrary holomorphic function, then $\tilde{\Phi} = \Phi + f + \overline{f}$ obeys $\partial\bar{\partial}\tilde{\Phi} = \partial\bar{\partial}\Phi$.*

To formalise things, let us define the Kähler manifold $\mathcal{K}$.

Definition. A Kähler manifold $\mathcal{K}$ is a complex manifold with a Kähler metric, which satisfies the following equivalent conditions:

1. The holonomy is in $U(N) \subset SO(2N)$.
2. The Kähler form is closed, i.e. $dk = (\partial + \bar{\partial})\left(ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}\right) = 0$.
3. The metric is locally of the form $g_{i\bar{j}} = g_{\bar{j}i} = \frac{\partial^2\Phi(z,\bar{z})}{\partial z^i \partial\bar{z}^{\bar{j}}}$ with a Kähler potential Φ.

4 The Calabi-Yau Manifolds

A Kähler manifold $\mathcal{K}$ – which by our above remark – admits an infinity of Kähler metrics. One may ask, is it possible to find a unique Kähler metric on $\mathcal{K}$? In other words, can one find on $\mathcal{K}$ a Kähler metric that has not $U(N)$ holonomy but $SU(N)$ holonomy? It is relatively easy to see that there is a topological obstruction to finding such a metric.

Calabi-Yau Metric

In fact, the spin connection $\hat{\omega}$ of a Kähler manifold is a $U(N)$ or $SU(N) \times U(1)$ gauge field. The $U(1)$ part of $\hat{\omega}$ is an abelian gauge field, which we may call $\hat{A}$. A metric on $\mathcal{K}$ of $SU(N)$ holonomy would be a Kähler metric such that $\hat{A}$ is a *pure gauge* or, in other words, such that the gauge-invariant field strength $\hat{F} = d\hat{A}$ vanishes. Such a metric exists only if $\hat{A}$ is topologically trivial, which requires that the associated field strength F has zero flux through any closed two-dimensional surface Σ in $\mathcal{K}$, $I(\Sigma) = \int_\Sigma \hat{F} = 0$.

Therefore, asking whether a Kähler manifold $\mathcal{K}$ admits a metric of $SU(N)$ holonomy amounts to asking whether it is possible to choose a Kähler potential so that the $U(1)$ part of the spin connection vanishes. In cohomology theory, the closed two form $\hat{F}$ defines an element of $H^2(\mathcal{K};\mathbb{R})$, the second de Rham cohomology group of $\mathcal{K}$ with real coefficients. This element[6] is called the **first Chern class** of $\mathcal{K}$, $c_1(\mathcal{K})$. A Kähler manifold such that $I(\Sigma) = 0$ for all Σ is said to have vanishing first Chern class, $c_1(\mathcal{K}) = 0$.

In 1957, E. Calabi [4] conjectured that a Kähler manifold $\mathcal{K}$ of $c_1(\mathcal{K}) = 0$ always admits a metric of $SU(N)$ holonomy. He also proved that there would be up to scaling a unique metric of $SU(N)$ holonomy for any choice of complex structure on $\mathcal{K}$ and the cohomology class of the Kähler form. Twenty years later, S.-T. Yau [5] proved the existence of this hypothetical metric by establishing a *global existence theorem* for the solutions of the nonlinear partial differential equations corresponding to Calabi's conjecture.

By virtue of this rather difficult theorem, metrics of $SU(N)$ holonomy correspond precisely to Kähler manifolds of $c_1 = 0$. This is a crucial simplification in our search for vacuum states of unbroken $\mathcal{N} = 1$ supersymmetry, since metrics of $SU(N)$ holonomy are extremely difficult to describe (none are known explicitly except in certain singular limits), but Kähler manifolds of $c_1 = 0$ can be found by qualitative methods, as we will see.

While it would hardly be possible to prove here the existence of metrics of $SU(N)$ holonomy on Kahler manifolds of $c_1 = 0$, we shall point out the following facts.

Claim. *A metric of $SU(N)$ holonomy is a Ricci-flat Kähler metric. (This is now called the Calabi-Yau metric[7].)*

Short proof. From the discussion of $SU(3)$ holonomy, it is true more generally that a manifold of $SU(N)$ holonomy admits a spinor field η that is covariantly constant, $\nabla_m \eta = 0$. Such a spinor field necessarily obeys $[\nabla_m, \nabla_n]\,\eta = 0$ or $R_{mnpq}\gamma^{pq}\eta = 0$, which in turn implies that $R_{mn} = 0$. Thus, a metric of $SU(N)$ holonomy is necessarily Ricci-flat. Since $SU(N) \subset U(N)$, a metric of $SU(N)$ holonomy is also Kähler. □

There is in fact a (longer) proof which concerns the spin connection $\hat{\omega}$.

Long proof. We shall prove our claim by showing that, up to a factor, the Ricci tensor of a Kähler manifold is the field strength $\hat{F}$ of the $U(1)$ part of $\hat{\omega}$, which we called $\hat{A}$. The embedding of $U(1)$ in $SO(2N)$ means that the $U(1)$ generator is precisely the complex structure $J^m{}_n$, whose nonzero components are $J^i{}_j = i\delta^i{}_j$, $J^{\bar{i}}{}_{\bar{j}} = -i\delta^{\bar{i}}{}_{\bar{j}}$.

[6] Strictly speaking, the closed two forms $\hat{F}$ are Chern forms; the Chern classes c_i are the cohomology classes of the Chern forms. When E is the holomorphic line bundle $T^{1,0}\mathcal{K}$, $c_i(E)$ is the Chern class of the manifold $\mathcal{K}$ and it is commonly denoted as $c_i(\mathcal{K})$. In particular, the first Chern class $c_1(\mathcal{K})$ of $\mathcal{K}$ is that of the so-called *canonical line bundle* of $\mathcal{K}$.

[7] To date, there are no explicitly known Ricci-flat Kähler metrics on any nontrivial compact Calabi-Yau manifolds (other than trivial cases of tori). This remains an important open problem.

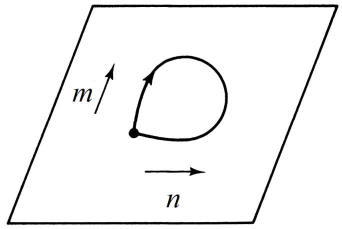

Figure 1: Under parallel transport of a vector V^p around a small loop in the mn plane, it changes by an amount $\delta V^p \sim R_{mn}{}^p{}_q V^q$.

Under parallel transport[8] around a small loop in the mn plane (Fig. 1), vectors are rotated by a matrix $R_{(mn)}$ whose pq matrix element is R_{mnpq}. The $U(1)$ part[9] of this matrix is $\hat{F}_{mn} = \mathrm{tr}\, JR_{(mn)} = R_{mnpq}J^{pq}$. This is the $U(1)$ part of the rotation undergone by tangent vectors that are parallelly transported around a small loop in the mn plane; it is the field strength $\hat{F}$ of $\hat{A}$. The nonzero components of $\hat{F}_{mn}$ are

$$\hat{F}_{i\bar{j}} = -\hat{F}_{\bar{j}i} = R_{i\bar{j}}{}^p{}_q J^q{}_p = iR_{i\bar{j}}{}^k{}_k - iR_{i\bar{j}}{}^{\bar{k}}{}_{\bar{k}}, \tag{4.1}$$

but

$$R_{i\bar{j}}{}^k{}_k = R_{i\bar{j}\bar{l}k} g^{\bar{l}k} = -R_{i\bar{j}k\bar{l}} g^{k\bar{l}} = -R_{i\bar{j}}{}^{\bar{l}}{}_{\bar{l}}. \tag{4.2}$$

Hence we have $\hat{F}_{i\bar{j}} = 2iR_{i\bar{j}}{}^k{}_k$. Comparing this to the Ricci tensor $R_{i\bar{j}} = R_i{}^k{}_{\bar{j}k}$ and the Kähler identity $R_i{}^k{}_{\bar{j}k} = -R_{i\bar{j}}{}^k{}_k$, we get finally the relation between the Ricci tensor $R_{i\bar{j}}$ and the $U(1)$ field strength $\hat{F}_{i\bar{j}}$:

$$\hat{F}_{i\bar{j}} = -2iR_{i\bar{j}} = -\hat{F}_{\bar{j}i}. \tag{4.3}$$

It follows from (3.6) that the affine connection can be written as $\Gamma^{\bar{i}}{}_{\bar{j}\bar{i}} = \partial_{\bar{j}} \ln\det g$. The Riemann tensor is

$$R^{\bar{i}}{}_{\bar{j}k\bar{l}} = \partial_k \Gamma^{\bar{i}}{}_{\bar{j}\bar{l}} \tag{4.4}$$

and the Ricci tensor is then

$$R_{\bar{j}k} = R^{\bar{i}}{}_{\bar{j}\bar{i}k} = -\partial_k \Gamma^{\bar{i}}{}_{\bar{j}\bar{i}} = -\partial_{\bar{j}} \partial_{\bar{k}} \ln\det g. \tag{4.5}$$

The conditions that $\hat{F}_{i\bar{j}} = 0$ and hence $R_{i\bar{j}} = 0$ are consequently that $\ln\det g = f(z^i) + \overline{f}(\bar{z}^{\bar{i}})$, where $f(z^i)$ is an arbitrary holomorphic function of z^i. Locally, by a holomorphic change of

[8] On a general Riemannian manifold $\mathcal{M}$ of dimension $2N$, the Riemann tensor R_{mnpq} is antisymmetric in p and q. Fixing m and n, it can be viewed as a $2N \times 2N$ antisymmetric matrix or in other words a generator of $SO(2N)$ which we may call $R_{(mn)}$, whose pq matrix element is R_{mnpq}. The well-known relation $[\nabla_m, \nabla_n]V^p = R_{mn}{}^p{}_q V^q$ shows that in parallel transport around a small loop, tangent vectors undergo rotation by the $SO(2N)$ matrices $R_{(mn)}$. For $\mathcal{M}$ of $U(N)$ holonomy, the matrices $R_{(mn)}$ are $U(N)$ matrices, embedded in $SO(2N)$. In our familiar complex basis, a $U(N)$ generator U satisfies $U_{ij} = U_{\bar{i}\bar{j}} = 0$. Hence the matrix $R_{(mn)}$ is a $U(N)$ matrix when $R_{mnij} = R_{mn\bar{i}\bar{j}} = 0$.

[9] Given an $SO(2N)$ generator U, its $U(1)$ part is $\mathrm{tr}\, JU = U^m{}_n J^n{}_m$.

coordinates $z^i \to z'^i(z^j)$, we can put f in an arbitrary form, say $f = 1/2$. The Ricci flat condition is hence locally

$$\ln \det g = 1. \tag{4.6}$$

Of course, g here must be a Kähler metric, described locally as $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi$, with Kähler potential Φ. So locally, in trying to find a Ricci-flat Kähler metric, we are trying to adjust Φ to obey the condition (4.6). □

Formally, we have the definition of a Calabi-Yau manifold.

Definition. A Calabi-Yau manifold (or a Calabi-Yau N-fold), $\mathcal{K}$, is a compact Kähler manifold, which satisfies the following equivalent conditions:

1. $\mathcal{K}$ has $SU(N)$ holonomy.
2. $\mathcal{K}$ has vanishing first Chern class ($c_1 = 0$).
3. $\mathcal{K}$ is Ricci flat.

In order for it to be valid in the noncompact case, additional boundary conditions at infinity need to be imposed.

Examples of Calabi-Yau Manifolds

Yau's theorem is very important, because on the one hand metrics with $SU(3)$ holonomy are very complicated (e.g. they can have no continuous symmetries) and none are known explicitly. On the other hand, Kähler manifolds of $c_1 = 0$ can be found by simple constructions.

The simplest Kähler manifolds[10] that we can consider are $\mathbb{C}^N$ and $\mathbb{CP}^N$, are not Calabi-Yau – the first because it is not compact and the second because its Ricci tensor is not zero. However, it turns out that some subspaces of $\mathbb{CP}^N$ are. A simple example of a Kähler manifold of $c_1 = 0$ is a hypersurface of degree $N+1$ in $\mathbb{CP}^N$. This is simply the set of zeros of some selected k homogeneous polynomials $P_1, \cdots, P_k$ of degrees $d_1, \cdots, d_k$, viz. if

$$\mathcal{K} = \left\{ z = (z_1, \cdots, z_{N+1}) \in \mathbb{CP}^N \mid P_i(z) = 0 \text{ for } i = 1, \cdots, k \right\} \tag{4.7}$$

with the identification

$$P_i(\lambda z_1, \cdots, \lambda z_{N+1}) = \lambda^{\ell_i} P_i(z_1, \cdots, z_{N+1}) \quad \text{and} \quad N+1 = \sum_{i=1}^{k} \ell_i, \tag{4.8}$$

then $\mathcal{K}$ is a Calabi-Yau manifold. It is indeed a Kähler manifold since the metric induced from $\mathbb{CP}^N$ is Kähler.

[10] The complex projective space $\mathbb{CP}^N$ is the complex space $\mathbb{C}^{N+1}$ with the origin removed, and and quotiented out by the identification $z_i \sim \lambda z_i$ for any nonzero complex number λ, i.e. all points along lines through the origin are identified.

The simplest example of a three-complex-dimensional Calabi-Yau manifold[11] (**Calabi-Yau three-fold**) from this family is obtained from $N = 4$ and $k = 1$, so that $\ell_i = 5$. To be specific, let us choose the following homogeneous polynomial of degree five[12]:

$$P = \sum_{\alpha=1}^{5} z_\alpha^5. \tag{4.9}$$

Its zeroes define a three-dimensional subset of $\mathbb{CP}^4$. The resulting Calabi-Yau manifold is called a **quintic hypersurface**, which has a $c_1 = 0$, and, by virtue of Yau's proof of the Calabi conjecture, admits a metric of $SU(3)$ holonomy.

This construction can be easily generalised. Let us consider k homogeneous polynomials of degree $d_1, \cdots, d_k$ in $\mathbb{CP}^{k+3}$. The subset defined by the simultaneous vanishing of all k equations is a three-complex-dimensional Kähler manifold. To see which of these manifolds admit metrics of $SU(3)$ holonomy, we must compute the first Chern class c_1, which can be represented by a two-form (this is just $\hat{F}$ of the $U(1)$ part of $\hat{\omega}$ discussed above). In addition, we can define the second and third Chern classes c_2 and c_3 – two topological invariants on these manifolds – that can be represented by a four-form and six-form, respectively. Let $c = 1 + c_1 + c_2 + c_3$ denote the **total Chern class**[13]. Then for the complete intersection of k polynomials discussed above, one can show that

$$c = \frac{(1+J)^{k+4}}{(1+d_1 J)\cdots(1+d_k J)} = 1 + \left(k + 4 - \sum_{i=1}^{k} d_i\right) J + \cdots, \tag{4.10}$$

where J is the two-form[14] obtained by normalising the Kähler form of $\mathbb{CP}^{k+3}$ so that its integral over any $\mathbb{CP}^1$ is unity and then projecting into the manifold defined by the vanishing of the polynomials. The ith Chern class is then the term proportional to J^i in the power series expansion on the right-hand side. Hence c_1 vanishes if and only if

$$\sum_{i=1}^{k} d_i = k + 4. \tag{4.11}$$

Since a linear subspace of $\mathbb{CP}^N$ is just $\mathbb{CP}^{N-1}$, we are only interested in solutions to (4.11) with all $d_i \geq 2$. There are then just five possibilities, i.e. manifolds obtained by the vanishing of

[11] We do not discuss manifolds such as $K3 \times T^2$ and T^6, whose holonomy is a proper subgroup of $SU(3)$ since they cannot give rise to chiral fermions if the gauge field configuration is supersymmetric.

[12] We may consider more general quintic equations in $\mathbb{CP}^4$, such as $\sum_\alpha z_\alpha^5 + \lambda z_1 z_2^4 + \mu z_1^2 z_2^3 + \cdots = 0$; the resulting manifolds are all diffeomorphic although they have different complex structures induced from $\mathbb{CP}^4$.

[13] The total Chern class is defined as

$$c = \det\left(I + \frac{i}{2\pi}\hat{F}\right) = \exp\left[\mathrm{tr}\left(\log\left(I + \frac{i}{2\pi}\hat{F}\right)\right)\right] = 1 + c_1 + \cdots + c_n, \quad c_p = (\frac{i}{2\pi})^p \det \hat{F},$$

As $\hat{F}$ is a 2-form, c_p is a $2p$-form and vanishes for $p > n$.

[14] Powers of J denote wedge products, so that $J^n = 0$ for $n > 3$.

1. a quintic equation in $\mathbb{CP}^4$,
2. a quartic and quadratic equation in $\mathbb{CP}^5$,
3. a pair of cubic equations in $\mathbb{CP}^5$,
4. a cubic and two quadratic equations in $\mathbb{CP}^6$, and
5. four quadratic equations in $\mathbb{CP}^7$.

The first condition above is for $k = 1$, that is precisely our earlier example of a quintic hypersurface in $\mathbb{CP}^4$.

We may use the notation $Y_{(N;\ d_1,\cdots,d_k)}$ to denote the manifold obtained by the vanishing of polynomials of degrees $d_1, \cdots, d_k$ in $\mathbb{CP}^N$. The five manifolds described above all have $c_1 = 0$ and are all simply connected and hence have no harmonic vectors. They also have exactly one harmonic two-form which is the *Kähler form*. Moreover, the second Stiefel-Whitney class ω_2 is the mod-2 reduction of c_1, we have $\omega_2 = 0$, implying that spinors can be consistently defined on these manifolds.

Remark (Spinors on six-manifolds of $SU(3)$ holonomy)**.** *Corresponding to the decomposition of tangent vector as a* $\mathbf{3} \oplus \overline{\mathbf{3}}$ *of* $SU(3)$*, the six gamma matrices* $\gamma^m, m = 1, \cdots, 6$ *are naturally split into creation and annihilation operators* a^{*i} *and* a_j*,* $i, j = 1, \cdots, 3$*, obeying*

$$\left\{a^{*i}, a^{*j}\right\} = \{a_i, a_j\} = 0, \quad \left\{a^{*i}, a_j\right\} = \delta^i{}_j. \tag{4.12}$$

Taking both chiralities together, the spinor representation of $SO(6)$ *is eight-dimensional. It consists of a Fock vacuum* $|\Omega\rangle$ *(with* $a_i\,|\Omega\rangle = 0$*), and the states*

$$\begin{aligned} |\overline{\Omega}^i\rangle &= a^{*i}\,|\Omega\rangle\,, \\ |\Omega_i\rangle &= \frac{1}{2}\epsilon_{ijk}a^{*j}a^{*k}\,|\Omega\rangle\,, \\ |\overline{\Omega}\rangle &= \frac{1}{6}\epsilon_{ijk}a^{*i}a^{*j}a^{*k}\,|\Omega\rangle\,. \end{aligned} \tag{4.13}$$

Since gamma matrices reverse chirality, the states of one chirality are $|\Omega\rangle$ *and* $|\Omega_i\rangle$ *while* $|\overline{\Omega}^i\rangle$ *and* $|\overline{\Omega}\rangle$ *have the opposite chirality. The covariantly constant spinors* η *are linear combinations of the* $SU(3)$ *singlets* $|\Omega\rangle$ *and* $|\overline{\Omega}\rangle$*.*

Remark (Topological condition of Yang-Mills fields)**.** *What are the choices of the Yang-Mills gauge fields that permit the existence of an unbroken* $\mathcal{N} = 1$ *supersymmetry? Recall that the vanishing of* $\delta\chi^a$ *(2.28) gives*

$$F^a_{mn}\gamma^{mn}\eta = 0. \tag{4.14}$$

By splitting the gamma matrices into creation and annihilation operators, we will have

$$F^a_{ij} = 0 = F^a_{\bar{i}\bar{j}}, \quad g^{i\bar{j}}F^a_{i\bar{j}} = 0. \tag{4.15}$$

Being trivial locally (rather like the equation for $U(3)$ holonomy), $F^a_{ij} = 0 = F^a_{\bar{i}\bar{j}}$ have essentially a topological impact – they assert that the gauge field is a holomorphic connection on a holomorphic vector bundle E. On a manifold with local complex coordinates z^i and $\bar{z}^j$, the equations $F^a_{ij} = 0 = F^a_{\bar{i}\bar{j}}$ have the general solution:

$$A_i = V^{-1} \frac{\partial}{\partial z^i} V, \tag{4.16}$$

where $V(z, \bar{z})$ is an arbitrary (not necessarily unitary) matrix valued function of $z, \bar{z}$.

On the other hand, the equation $g^{i\bar{j}} F^a_{i\bar{j}} = 0$ is highly non-trivial, somewhat like the condition for $SU(3)$ as opposed to $U(3)$ holonomy. Locally, it is a non-linear partial differential equation for the unknown V. Uhlenbeck and Yau have proved [9] the global existence and uniqueness of the solution of this equation, at least for gauge group SU(N), under a mild assumption about E – it must be a stable bundle. Therefore, the solutions of (4.15) can be classified topologically.

In addition to the quintic hypersurface, there is one other known example of a simply connected Calabi-Yau three-fold that can be constructed with toric geometry; details can be found in [6] Section 3. While several thousand inequivalent Calabi-Yau three-folds have been obtained, the classification of Calabi-Yau three-folds is not yet complete to date [10]. More general Calabi-Yau manifolds, including noncompact ones, can also be found by looking at intersections of numerous constraints in higher dimensional projective and weighted projective spaces, and products thereof.

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