An Introduction to Planar AdS_5/CFT_4 Integrability

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You are a victim of your own neural architecture
which doesn't permit you to imagine anything outside of three dimensions.
Even two dimensions.
People know they can't visualise four or five dimensions,
but they think they can close their eyes
and see two dimensions.
But they can't.

Leonard Susskind

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Abstract

The four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) theory serves as a toy model with rich symmetries in the attempts to solve the hadronic physics described by Quantum Chromodynamics (QCD). It appears that $\mathcal{N} = 4$ SYM can be solved exactly in the planar limit. The miracle which leads to the solution of $\mathcal{N} = 4$ SYM is generally called integrability. Through the AdS/CFT correspondence which relates certain pairs of models, similar integrable structures are found in the type IIB superstring theory on the $AdS_5 \times S^5$ background. Proving such duality is not a simple task, but the discovery of integrability in the planar AdS/CFT has allowed us to reach immense progresses in understanding and confirming the duality. This dissertation presents a pedagogical introduction to the concepts of integrability and of both sides of the AdS_5/CFT_4 correspondence. Classical and quantum integrability of the dual theories are also briefly discussed with some astonishing established results.

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Chapter 1

Introduction

The relation between gauge theories and string theory is a long-standing unsolved problem in theoretical physics. Understanding this relation is a great step to handle gauge theories at strong coupling and consistently include gravity into the picture of quantum field theory. This introductory chapter presents the motivations to study the two essential concepts which will be explored in this dissertation, i.e. integrability and anti-de Sitter/conformal field theory (AdS/CFT) correspondence.

Integrability

Soon after Newton's equations were formulated and the Kepler problem was solved exactly by Newton himself, finding *exact solutions* for other nontrivial cases has been a major interest. Nevertheless, only a handful of problems could be treated exactly. In the 1800s Liouville refined the notion of *integrability* for Hamiltonian systems, providing a general framework for solving particular dynamical systems by *quadratures*. However, it was not until a decade later that Gardner, Green, Kruskal, and Miura invented the *classical inverse scattering method* that successfully solve the Korteweg-de Vries (KdV) equation[1] [51].

The quantum analogue of the inverse scattering method, known as *algebraic Bethe ansatz*[2], was established in 1979 by the Leningrad - St. Petersburg school, led by Ludwig Faddeev, with Korepin, Kulish, Reshetikhin, Sklyanin, Semenov Tian-Shansky, Takhtajan,

[1] KdV equation – an exactly solvable nonlinear partial differential equation (PDE) – models the surface waves on shallow water. It is well-known as a prototype of an exactly solvable model.

[2] The German word *ansatz* means an approach or method of making an educated guess for the solution that is later verified by its results.

and many others. This systematic approach to integrable[3] quantum mechanical systems has the power of unifying integrable quantum field theories and lattice spin systems in a single mathematical framework, and reformulate $(1+1)$-dimensional problems in Drinfeld and Jimbo's theory of *quantum groups* [38].

Although *integrability* and *solvability* often go hand-in-hand making integrable theories so appealing, it is probably worthwhile mentioning that the two terms are not the same. There exist exactly solvable systems which are not integrable, as well as integrable systems which one cannot solve to the very end. *Solvability* depends on one's ability and computational power, whereas *integrability*[4] rather refers to the conservation laws and property of a system to exhibit regular (quasi-periodic) or chaotic behaviour. This indeed provides general mathematical methods for an exact solution of a problem.

Why integrability?

Integrability is a phenomenon largely restricted to $(1+1)$-dimensional systems. One can ask, what is good in $(1+1)$-dimensional models, when our spacetime is $(3+1)$-dimensional. There are several particular answers [43] provided by L. Faddeev to this query.

(i) The toy models in $(1+1)$ dimension teach us about the realistic field-theoretical models in a nonperturbative way. Indeed phenomena such as renormalisation, asymptotic freedom, dimensional transmutation (i.e. the appearance of mass via the regularisation parameters) hold in integrable models and can be described exactly.

(ii) There are numerous physical applications to a multitude of widely differing problems in the condensed matter physics, superconductivity models, nonlinear optics, and mathematical physics.

(iii) The formalism of integrable models showed several times to be useful in the modern string theory, in which the world-sheet is 2-dimensional. Moreover, the conformal field theory models are special massless limits of integrable models.

(iv) The theory of integrable models teaches us about new phenomena, which were not appreciated in the previous developments of Quantum Field Theory, especially in connection with the mass spectrum.

[3] The adjective *integrable* stems from a paper of Faddeev and Zakharov of 1971 where the KdV equation was shown to be interpreted as an integrable (though infinite–dimensional) Hamiltonian system.

[4] Integrability in the quantum mechanical sense implies specific properties of the scattering theory and of the spectrum.

(v) Working with the integrable models is a delightful pastime. They proved also to be very successful tool for the educational purposes.

AdS/CFT Correspondence

In the last twenty years, string/gauge duality (or more precisely AdS/CFT duality), and relevant holographic[5] concepts, have become a dominant area in fundamental physics. The basic idea is that a gravitational theory[6], defined on a $(D+1)$-dimensional spacetime (*the bulk*) is equivalent to a gauge theory (CFT), defined on a D–dimensional spacetime (*the bulk's boundary*). This equivalence is expressed by a *correspondence* or *dictionary* between the observables and quantities of one theory and those of the other.

Why AdS/CFT?

Much of the interest in the duality arises from the facts that:

(i) The duality dictionary is a good source for *holographers* to translate two distinct concepts into each other. For instance, a direction in the bulk spacetime is translated as the direction of the renormalisation group flow in the boundary theory.

(ii) Each theory is useful to help solve problems in the other. In particular, a strong coupling regime in one theory, where problems are difficult to solve by perturbative methods, can be translated into the weak coupling regime of other theory, where problems are easier to solve, and vice versa.

---

[5]The physics of the bulk adequately encoded in the boundary is reminiscent of *holography*: hence the use of the term.

[6]A gravitational theory does not literally mean Einstein gravity, but is rather a theory on dynamical spacetime, for example, string theory on anti-de Sitter (AdS) spactime.

Outline

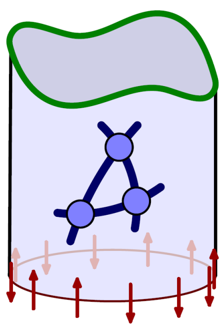
Integrability in AdS/CFT relates two theories living in different dimensions with different fields content to an integrable system in one dimension (in particular, spin chain). This dissertation focuses on the emergence of integrability in the AdS_5/CFT_4 system, and is organised into three main parts.

In **Part I**, we will convey a qualitative understanding of integrability, i.e. how and why it works in classical and quantum regimes. In order to supplement the main text, some background aspects of the superconformal algebra $\mathfrak{psu}(2,2|4)$ will be presented in the appendix section.

In **Part II**, we introduce the best understood example in CFT – the maximally supersymmetric $SU(N)$ Yang-Mills theory in $(3+1)$ dimensions (or the so-called $\mathcal{N}=4$ SYM). We will then discuss the emergence of integrability in the $\mathcal{N}=4$ SYM theory to one loop order.

In **Part III**, we first explore the dual description of AdS_5/CFT_4 in terms of type IIB superstring theory on $AdS_5 \times S^5$ spacetime. We shall also outline the principle behind AdS_5/CFT_4 which conjectures the equivalence of the $\mathcal{N}=4$ SYM theory and $AdS_5 \times S^5$ IIB superstring theory, albeit AdS/CFT still lacks a formal proof. To conclude this part, we investigate the integrability of classical IIB superstring theory on $AdS_5 \times S^5$ spacetime. This will indeed prove some features that both theories have in common, and hence confirming the AdS_5/CFT_4 conjecture.

Several important remarks will be summarised in the last chapter. In addition to that, a list of topics which were uncovered in the present dissertation and interesting open questions will be presented in the concluding chapter.



Integrability of AdS/CFT. This illustration is adapted from [14].

Part I

Integrable Systems

Chapter 2

Classical Integrable Systems

Korteweg-de Vries (KdV) model.

2.1 Hamiltonian Dynamical Systems

We start by considering a classical dynamical system in Hamiltonian formulation where a point particle with mass m moves in a potential $U(q)$, where $q = (q^1, \ldots, q^n)$, and has the momentum $p = (p_1, \ldots, p_n)$, where $p_i = mq^i, \quad i = 1, 2, \ldots, n$. The phase space $\mathcal{M}$ of dimension $2n$ is then defined by a set of coordinates q^i and momenta p_i. Having the Hamiltonian[7] of the system,

$$H = -\frac{p^2}{2m} + U(q), \tag{2.1}$$

the Newton equations that describes the motion of the particle,

$$m\ddot{q}^i = -\frac{\partial U}{\partial q^i}. \tag{2.2}$$

can be rewritten as two first-order differential equations,

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.3}$$

[7] A Hamiltonian H can be thought of as a function mapping the phase space $\mathcal{M}$ to a real space, $H : \mathcal{M} \longrightarrow \mathbb{R}$.

Equation (2.3) are the Hamilton's equations of motion which a solution of the system, i.e. a curve $(q^i(t), p_i(t))$ in phase space obeys. Further, we can rewrite the Hamilton's equations as

$$\dot{x} = \Omega \cdot \nabla H, \tag{2.4}$$

where we have introduced two $2n$-dimensional vectors x and ∇H, and a $2n \times 2n$ nonsingular and skew-symmetric symplectic matrix Ω,

$$x = \begin{pmatrix} q^i \\ p_i \end{pmatrix}, \quad \nabla H = \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{2.5}$$

The symplectic matrix Ω motivates us to define the Poisson brackets[8] on the space of differentiable functions on $\mathcal{M}$, $\mathcal{F}(\mathcal{M})$,

$$\{f, g\}(x) = (\nabla f, \Omega \nabla g) = \Omega^{\alpha\beta} \partial_\alpha f \partial_\beta g = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \quad \alpha, \beta = 1, 2. \tag{2.6}$$

The Poisson brackets allow us to write the evolution equation of a function $f(q, p, t)$, evaluated on a solution $(q^i(t), p_i(t))$, in a compact fashion,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} - \{H, f\}. \tag{2.7}$$

For a time-independent Hamiltonian, $\frac{\partial H}{\partial t} = 0$, the function $H(q, p)$ is an *integral of motion* or a *conserved quantity*, i.e.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} - \{H, H\} = 0. \tag{2.8}$$

Therefore, the solutions (or motions of the system) are constrained to a hypersurface[9] of $\mathcal{M}$ with $H(q, p) = E$, where E is constant energy.

---

[8]The Poisson brackets are defined by the canonical relations of its basis elements $\{x^i, x^j\} = \Omega^{ij}$ which can be written as $\{q^i, p_j\} = \delta^i_j$ and $\{q^i, q^j\} = \{p_i, p_j\} = 0$. The brackets are anti-symmetric, obey the Jacobi identity for arbitrary phase space functions f, g, k, and satisfies the Leibnitz rule,

$$\begin{aligned} \{f, g\} &= -\{g, f\}, \\ \{f, \{g, k\}\} + \{g, \{k, f\}\} + \{k, \{f, g\}\} &= 0, \\ \{f, gk\} &= \{f, g\}k + g\{f, k\}. \end{aligned}$$

[9]A hypersurface is a smooth manifold of $n - 1$ dimension, which is embedded in an n-dimensional space and defined by a single implicit equation (at least locally).

The Hamilton's equations (2.3) can now be written as

$$\left\{H, q^i\right\} = \frac{\partial H}{\partial p_i}, \quad \{H, p_i\} = -\frac{\partial H}{\partial q^i}. \tag{2.9}$$

Since these are coupled nonlinear differential equations it is usually impossible to find the exact solutions of the Hamilton's equations for arbitrary initial data $q^i(0), p_i(0)$. However, there are special cases where analytic solutions exist, which are closely related to the existence of integrals of motion (conserved quantities) which remain constant along the particle trajectories.

For any (time-independent) integrals of motion $f_j(q,p)$, we have $\frac{df_j}{dt} = 0$ if and only if $\{H, f_j\} = 0$. Thus the integrals of motion correspond to functions constrained to $f_j = c_j$ constant on an even lower dimensional hypersurface, called a *level set*, $M_c := \{x \in M : f_j(x) = c_j\}$. Note that the Hamiltonian H is also in the level set where one may identify $f_1 = H$. This leads us to the Liouville theorem.

Theorem 2.1 (Liouville theorem)**.** *A system with* $2n$*-dimensional phase space* $\mathcal{M}$ *is (Liouville) integrable if it is solvable by quadratures. In other words, it is sufficient to solve the Hamilton's equations by solving a finite number of algebraic equations and integrals, instead of solving differential equations, i.e. the equations of motion.*

The theorem essentially implies that a (Liouville) integrable system must have n independent, everywhere differentiable, integrals of motion f_i which are *in involution* or Poisson commute with each other: $\{f_i, f_j\} = 0$.

2.2 Lax Pairs and Classical r-matrix

We shall introduce two prominent structures for integrable models that will help us solve a model. These will appear later in the context of integrable field theories (see section 2.3). Using the two structures, the Poisson bracket can be recast in a suitable form that displays the symmetry structure of a model.

Lax pair. Suppose a pair of square matrices L, M whose entries are functions of phase space, i.e. $L = L(q,p), M = M(q,p)$, such that the Hamilton's equations can be recast in the *Lax equation*

$$\dot{L} = [M, L], \tag{2.10}$$

where $\cdot$ denotes the total time derivative, then the two matrices L, M are said to form a *Lax pair* or *Lax representation*[10]. The solution of the Lax equation is

$$L(t) = g(t)L(0)g(t)^{-1}, \tag{2.11}$$

with an invertible matrix $g(t)$ that also depends on the phase-space variables and is determined from the equation

$$M(t) = \dot{g}g(t)^{-1}. \tag{2.12}$$

To solve the eigenvalue problem $LX = \lambda X$ at time t, it is the same to solve the problem at time 0 where L is known, and to propagate the solution with the conditions

$$\begin{aligned} \lambda(t) = \lambda(0) \quad \text{(no change in spectrum)}, \\ \partial_t X = MX, \end{aligned} \tag{2.13}$$

where X is an eigenfunction of L. Notice that the Lax pair is *not unique*, as there is at least a gauge freedom

$$L \longrightarrow gLg^{-1}, \quad M \longrightarrow gMg^{-1} + \dot{g}g^{-1}. \tag{2.14}$$

The non-uniqueness of L can be shown by taking the time-derivative of L,

$$\dot{L} = \dot{g}Lg^{-1} + g[M, L]g^{-1} - gLg^{-1}\dot{g}g^{-1} = \left[gMg^{-1} + \dot{g}g^{-1}, gLg^{-1}\right] = [M, L]. \tag{2.15}$$

We can immediately see that the trace of power of the matrix L generates a tower/set of integrals of motion f_n,

$$f_n \equiv \operatorname{tr} L^n \quad \forall n \geqslant 0, \tag{2.16}$$

which are indeed conserved by simply opening up the commutator and using the cyclicity of the trace

$$\dot{f}_n = n \operatorname{tr}(L^{n-1}\dot{L}^n) = n \operatorname{tr}(L^{n-1}[M, L]) = \operatorname{tr}[M, L^n] = 0. \tag{2.17}$$

Therefore, the integrals f are functions of the eigenvalues of the matrix L.

A simple example to illustrate the integrals of motion will be a harmonic oscillator

[10] Lax pairs were introduced by Peter Lax in a seminal paper in 1968 to discuss solitons in continuous media [69]. In layman's terms, a soliton or solitary wave is caused by a cancellation of nonlinear and dispersive effects in a medium. Its shape maintains while propagating at a constant velocity in the medium. The first account of solitons goes back to John Scott Russell in 1834.

with frequency ω. The Hamiltonian of the harmonic oscillator is

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2. \tag{2.18}$$

One can choose

$$L = \begin{pmatrix} +p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -\frac{1}{2}\omega \\ +\frac{1}{2}\omega & 0 \end{pmatrix}. \tag{2.19}$$

The Lax equation

$$\begin{aligned} \begin{pmatrix} \dot{p} & \omega\dot{q} \\ \omega\dot{q} & -\dot{p} \end{pmatrix} &= \begin{pmatrix} 0 & -\frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{pmatrix}\begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix} - \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}\begin{pmatrix} 0 & -\frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix} \end{aligned} \tag{2.20}$$

gives the equations of motion of the harmonic oscillator

$$\dot{p} = -\omega^2 q, \quad \dot{q} = p, \tag{2.21}$$

and the resulting integrals of motion read

$$\begin{aligned} f_1 &= 0, \\ f_2 &= 2p^2 + 2\omega^2 q^2 = 4H, \\ f_3 &= 0, \\ f_4 &= 2\left(p^2 + \omega^2 q^2\right)^2 = 8H^2, \\ &\dots . \end{aligned} \tag{2.22}$$

f_1 and f_3 are trivial and can be ignored. The first and only independent integral of motion f_2 is the Hamiltonian $H = \frac{1}{4}\operatorname{tr} L^2$. The higher even powers are merely powers of the Hamiltonian which are not independent integrals of motion.

Classical r-matrix. Since the eigenvalues of L are independent of time, the evolution of the system is called *isospectral*. To ensure that the conserved eigenvalues of L are in involution, we consider the general form of the Poisson bracket between the matrix elements of L and suppose that L is diagonalisable, i.e.

$$L = U\Lambda U^{-1}, \tag{2.23}$$

we obtain

$$
\begin{aligned}
\{L_1, L_2\} &= \left\{U_1\Lambda_1 U_1^{-1}, U_2\Lambda_2 U_2^{-1}\right\} \\
&= \underbrace{\{U_1, U_2\}\,\Lambda_1 U_1^{-1}\Lambda_2 U_2^{-1}}_{1} + U_2\,\{U_1, \Lambda_2\}\,\Lambda_1 U_1^{-1} U_2^{-1} \\
&\quad - \underbrace{U_2\Lambda_2 U_2^{-1}\,\{U_1, U_2\}\,U_2^{-1}\Lambda_1 U_1^{-1}}_{2} + U_1\,\{\Lambda_1, U_2\}\,U_1^{-1}\Lambda_2 U_2^{-1} \\
&\quad - U_2\Lambda_2 U_2^{-1} U_1\,\{\Lambda_1, U_2\}\,U_2^{-1} U_1^{-1} - \underbrace{U_1\Lambda_1 U_1^{-1}\,\{U_1, U_2\}\,U_1^{-1}\Lambda_2 U_2^{-1}}_{3} \\
&\quad - U_1\Lambda_1 U_2 U_1^{-1}\,\{U_1, \Lambda_2\}\,U_1^{-1} U_2^{-1} + \underbrace{U_1\Lambda_1 U_1^{-1} U_2\Lambda_2 U_2^{-1}\,\{U_1, U_2\}\,U_1^{-1} U_2^{-1}}_{4},
\end{aligned}
\tag{2.24}
$$

where we have assumed that the eigenvalues commute $\{\Lambda_1, \Lambda_2\} = 0$. Introducing

$$
k_{12} = \{U_1, U_2\}\,U_1^{-1}U_2^{-1}, \qquad q_{12} = U_2\,\{U_1, \Lambda_2\}\,U_1^{-1}U_2^{-1}, \quad q_{21} = U_1\,\{U_2, \Lambda_1\}\,U_1^{-1}U_2^{-1},
\tag{2.25}
$$

we can write

$$
\begin{aligned}
\{L_1, L_2\} &= \underbrace{k_{12}L_1L_2}_{1} + q_{12}L_1 - \underbrace{L_2k_{12}L_1}_{2} - q_{21}L_2 \\
&\quad + L_2q_{21} - \underbrace{L_1k_{12}L_2}_{3} - L_1q_{12} + \underbrace{L_1L_2k_{12}}_{4} \\
&= \underbrace{[k_{12}L_2 - L_2k_{12}, L_1]}_{1,2,3,4} + [q_{12}, L_1] - [q_{21}, L_2] \\
&= \frac{1}{2}\left[[k_{12}, L_2], L_1\right] - \frac{1}{2}\left[[k_{21}, L_1], L_2\right] + [q_{12}, L_1] - [q_{21}, L_2] \\
&= [r_{12}, L_1] - [r_{21}, L_2],
\end{aligned}
\tag{2.26}
$$

where we have introduced the *r-matrix*

$$
r_{12} = q_{12} + \frac{1}{2}\,[k_{12}, L_2]\,.
\tag{2.27}
$$

For the Jacobi identity to hold for the bracket, we impose the following condition:

$$
\begin{aligned}
&[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \\
&[L_2, [r_{13}, r_{21}] + [r_{23}, r_{21}] + [r_{23}, r_{31}] + \{L_3, r_{21}\} - \{L_1, r_{23}\}] + \\
&[L_3, [r_{31}, r_{12}] + [r_{21}, r_{32}] + [r_{31}, r_{32}] + \{L_1, r_{32}\} - \{L_2, r_{31}\}] = 0.
\end{aligned}
\tag{2.28}
$$

Solving this equation for r is equivalent to classifying integrable systems. If r_{12} is a constant, independent of the dynamical variables, and antisymmetric, $r_{12} = -r_{21}$, then a sufficient condition for the Jacobi identity to be satisfied is

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0. \tag{2.29}$$

This equation is called the **classical Yang-Baxter equation (CYBE)**. Notice that the r-matrix is not uniquely defined by the equation, much like for the Lax pair.

Matrices L_1 and L_2 can be regarded as elements of some matrix algebra $\mathfrak{g}$: $L_1, L_2 \in \mathfrak{g} = \mathrm{End}(V)$. Equation (2.26) is defined on the tensor product space $\mathfrak{g} \otimes \mathfrak{g}$ of two matrices, i.e. $L_1 := L \otimes 1, L_2 := 1 \otimes L$, and the classical r-matrix r_{12} is a particular element of this space whose matrix entries are functions of phase-space variables. Also, $r_{21} := P\,(r_{12})$ denotes the permutation of the two spaces for the r-matrix.

The most interesting case (present in integrable field theories) is when the Lax pairs depend analytically on an auxiliary complex variable u, called the **spectral parameter**, which is not directly related to the dynamics of the model. This means that we can find in some cases a family of Lax pairs $L_1(u), L_2(u)$, parameterised by u, such that the equations of motion are equivalent to the condition

$$\dot{L}_1(u) = [L_2(u), L_1(u)] \quad \forall u \in \mathbb{C}. \tag{2.30}$$

There are more constraints on this functional equation compared to the Lax equation without spectral parameter. This comes handy in mechanical systems with infinitely many degrees of freedom whose equations of motion could be formulated by a finite-dimensional Lax pair. The expansion of the $f_n(u)$ in u can yield infinitely many independent integrals of motion which are needed for integrability of such models. By making use of the analytical structure of $L(u)$ in the complex spectral parameter u we can transform a mechanical system to a problem of complex analysis that ease the investigation of the dynamics.

The classical r-matrix can also be generalised to admit spectral parameters. It is now a function $r_{12}\,(u_1, u_2)$ of two spectral parameters $u_1, u_2 \in \mathbb{C}$ associated to each of the two related Lax matrices. The Lax equation now reads

$$\{L_1\,(u_1)\,, L_2\,(u_2)\} = [r_{12}\,(u_1, u_2)\,, L_1\,(u_1)] - [r_{21}\,(u_2, u_1)\,, L_2\,(u_2)]\,. \tag{2.31}$$

2.3 Integrable Classical Field Theory

We now generalise the classical finite-dimensional systems to $(1+1)$-dimensional fields $\psi(t,x)$. If we take a time slice, then a field can be Taylor or Fourier expanded leading to infinitely many independent coefficients. Therefore, the phase space for fields is infinite-dimensional, and thus integrability requires infinitely many conserved quantities in involution. When the number of degrees of freedom becomes infinite, the definition of integrability, based on the Liouville theorem, is unclear. We will be satisfied with the availability of efficient constructive methods for solutions. Whether or not a model is formally integrable will be of little concern.

Although most field theory models are non-integrable, there are several well-known models that are integrable, such as Korteweg-de Vries (KdV) equation, nonlinear Schrödinger equation, sine-Gordon equation, and classical Heisenberg magnet (Landau-Lifshitz equation). Among the important boundary conditions for these models are

- infinite spatial extent with rapidly decaying fields (or derivatives): $\psi(x,t) \to 0$ as $|x| \to \infty$,
- closed or periodic boundary conditions: $\psi(x+2\pi,t) = \psi(x,t)$,
- open boundaries with Dirichlet or Neumann conditions: $\psi =$ constant or $\psi' = 0$.

For a general field theory, we introduce a spectral-parameter dependent Lax pair that serves as a local object in field theory, which is called the **Lax connection** $U(x,t,u)$ and $V(x,t,u)$. The spectral parameter u here is a continuous parameter whose Taylor expanding the Lax connection in u will lead to an infinite tower of conserved quantities. The Lax connection satisfies the flatness or *zero-curvature* condition for all u (which is equivalent to the Euler-Lagrange equations),

$$\partial_t U - \partial_x V + [U,V] = 0. \tag{2.32}$$

This condition arises from the consistency condition for the auxiliary linear problem,

$$\begin{aligned} \partial_x \psi &= U(x,t,u)\psi, \\ \partial_t \psi &= V(x,t,u)\psi. \end{aligned} \tag{2.33}$$

as can be seen by applying ∂_t to the first equation, ∂_x to the second equation,

$$\begin{aligned} \partial_t\partial_x\psi &= \partial_t U(x,t,u)\psi + U(x,t,u)\partial_t\psi = \left[\partial_t U(x,t,u) + U(x,t,u)V(x,t,u)\right]\psi, \\ \partial_x\partial_t\psi &= \partial_x V(x,t,u)\psi + V(x,t,u)\partial_x\psi = \left[\partial_x V(x,t,u) + V(x,t,u)U(x,t,u)\right]\psi, \end{aligned} \tag{2.34}$$

and subtracting the two. If we introduce a gauge field $\mathcal{L}_\alpha$ with components $\mathcal{L}_x = U, \mathcal{L}_t = V$, then we have the condition of vanishing of the curvature of $\mathcal{L}_\alpha$,

$$F_{\alpha\beta}(\mathcal{L}) \equiv \partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha - [\mathcal{L}_\alpha, \mathcal{L}_\beta] = 0. \tag{2.35}$$

The one-parameter family of the flat connections allows us to define the **monodromy matrix** $T(u)$ which is the path-ordered exponential of the Lax component $U(u)$,

$$T(u) = \mathcal{P} \exp \int_{s_-}^{s_+} \mathrm{d}x U(u) = \mathcal{P} \exp \int_0^{2\pi} \mathrm{d}x U(u), \tag{2.36}$$

where $\mathcal{P}$ denotes a path-ordering operator with greater x to the left, s_- and s_+ are two points on the spatial line, taken to be 0 and 2π here. As the Lax connection is really a *connection*[11], $T(u)$ can be thought of as implementing a parallel transport along a compact segment $[0, 2\pi]$.

Let us derive the time evolution equation for this matrix. We get

$$\begin{aligned}
\partial_t T(u) &= \int_0^{2\pi} \mathrm{d}x\, \mathcal{P}\, e^{\int_x^{2\pi} \mathrm{d}y\, U}\, (\partial_t U)\, \mathcal{P}\, e^{\int_0^x \mathrm{d}y\, U} \\
&= \int_0^{2\pi} \mathrm{d}x\, \mathcal{P}\, e^{\int_x^{2\pi} \mathrm{d}y\, U}\, (\partial_x V + [V, U])\, \mathcal{P}\, e^{\int_0^x \mathrm{d}y\, U} \\
&= \int_0^{2\pi} \mathrm{d}x\, \partial_x \left(\mathcal{P}\, e^{\int_x^{2\pi} \mathrm{d}y\, U} V\, \mathcal{P}\, e^{\int_0^x \mathrm{d}y\, U} \right) \\
&= V(2\pi, t, u) T(u) - T(u) V(0, t, u) \\
&= [V(2\pi, t, u), T(u)],
\end{aligned} \tag{2.37}$$

where we used the flatness of $\mathcal{L}_\alpha \equiv (U, V)$ in the second line and assumed the periodic boundary conditions on the fields, $V(0, t, u) = V(2\pi, t, u)$, in the last line. This implies that the trace of $T(u)$, called the **transfer matrix**,

$$\mathfrak{t}(u) \equiv \operatorname{tr} T(u) \tag{2.38}$$

is conserved for all u. One can easily see that the eigenvalues of $T(u)$ generate a family of conserved charges Q_n upon Taylor-expanding in u if $\mathfrak{t}(u)$ is analytic near the origin,

$$\mathfrak{t}(u) = \sum_{n=0}^{\infty} Q_n u^n, \quad \partial_t Q_n = 0, \forall n \geqslant 0, \tag{2.39}$$

[11] A connection specifies how tensors are transported parallelly along a curve.

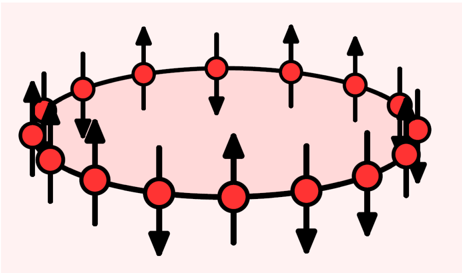
or Laurent-expanding around some point $u = \infty$,

$$\mathfrak{t}(u) = \sum_{n=0}^{\infty} \frac{Q_n}{u^n}, \quad \partial_t Q_n = 0, \forall n \geqslant 0. \tag{2.40}$$

Therefore, the monodromy matrix encodes the spectral properties of the model and is the starting point for the construction of the integrable structure. Both Lax connection and monodromy matrix are used in the *inverse scattering method* (a nonlinear analogue of the Fourier transform) to recover a potential from the scattering matrix in integrable models, as opposed to the direct scattering method of finding the scattering matrix from the potential. We recommend to the serious reader to study the inverse scattering method in [103].

Chapter 3

Quantum Integrable Systems



A closed Heisenberg $XXX_{1/2}$ spin chain.

3.1 Quantum Integrability

A common procedure to quantise an integrable dynamical system is to replace a phase space by a Hilbert space, and the variables q, p, the Hamiltonian H, and the Poisson bracket by

$$q \mapsto \hat{q}, \quad p \mapsto \hat{p}, \quad H \mapsto \hat{H}, \quad \{\ \ ,\ \ \} \mapsto \frac{\mathrm{i}}{\hbar}[\ \ ,\ \ ] \tag{3.1}$$

where all the operators are Hermitian operators and $[\ \ ,\ \ ]$ denotes a commutator. A straightforward definition of quantum integrability will be

Definition 3.1. A system is *quantum integrable* if

$$\left[\hat{H}, \hat{O}_i\right] = 0, \quad \left[\hat{O}_i, \hat{O}_j\right] = 0 \qquad \forall i, j \tag{3.2}$$

where operators $\hat{O}_i, \hat{O}_j$ for $i, j = 1, 2, \ldots, n$ can be diagonalised simultaneously with $\hat{H}$.

This definition of quantum integrability is, however, a bit sketchy. It is only valid for a sequence of consecutive two-body scattering processes, described by the non-diffractive part of the n-particle wavefunction in the asymptotic region,

$$\begin{aligned}\Psi_{\text{ asymp }}\left(x_1, x_2, \ldots, x_n\right) \sim & \sum_{\sigma \in \Pi_n} \Psi(\sigma) \exp i\left(p_{\sigma(1)} x_1+\ldots+p_{\sigma(n)} x_n\right) \\ & +\Psi_{\text{ difractive }}\left(x_1, x_2, \ldots, x_n\right) \quad \text{for} \quad x_1 \ll x_2 \ll \ldots \ll x_n,\end{aligned} \tag{3.3}$$

where σ denotes an element of the permutation group Π_n. The diffractive part contributes to the state of the system after three- and many-body processes. Consequently, the non-diffractive part of the wavefunction is fully determined by the two-body S-matrix. This fact allows for a more precise definition of the quantum integrability, i.e.

Definition 3.2. A quantum system that supports scattering is integrable if and only if the scattering of the particles is non-diffractive,

$$\Psi_{\text{diffractive}}\left(x_1, x_2, \ldots, x_n\right)=0 \quad \forall n. \tag{3.4}$$

This indeed implies that no particles are being created or annihilated in a scattering process; see also the discussion in [85].

To solve the model we need to find the resulting spectrum of $\hat{H}$ and $\hat{O}_i$. Unfortunately there is no quantum analogue of Liouville theorem but the spectrum can often be obtained exactly using a set of techniques known as the *Bethe ansatz*.

The term “Bethe ansatz” is simply a code name for a wavefunction with a specific structure, much as the Hartree, Hartree-Fock and Slater-Jastrow wavefunctions denote other kinds of structure. Although the structure of Bethe ansatz is well-defined in quantum mechanics, it does persist in the classical systems. There are several different constructions of Bethe ansatz:

- *Coordinate Bethe ansatz.* This method was originally put forward by Hans Bethe in 1931 to solve the spin-$\frac{1}{2}$ XXX Heisenberg model of magnetism [25] (see section 3.3).

- *Algebraic Bethe ansatz.* The coordinate Bethe ansatz totally obscures why a given Hamiltonian is integrable despite being very physical and widely applicable. It was realised in the late 1970’s and early 1980’s that the coordinate Bethe ansatz can be formulated in a rather systematic way to solve more general classes of integrable models. The technique is known as the algebraic Bethe ansatz which can solve exactly the spectral problems connected with the Yang-Baxter algebra. It is also known as the quantum inverse scattering method [44].

- *Functional Bethe ansatz*. The algebraic Bethe ansatz is only applicable if there exists a pseudo-vacuum. It certainly fails for models like the Toda chain, which has no pseudo-vacuum. In 1990, E. Sklyanin [96] devised another powerful technique – the functional Bethe ansatz – which has a transparent connection with the traditional separation of variables.

- *Nested Bethe ansatz*. Solving models with internal degrees of freedom proved to be very hard, because scattering involves changes of the internal states of scatters. The generalization of the Bethe ansatz to this problem was eventually solved by C. N. Yang and M. Gaudin by means of the so-called nested Bethe ansatz [52].

- *Asymptotic Bethe ansatz*. Most of the finite volume integrable systems cannot be solved by the Bethe ansatz techniques. This method provides the leading finite-size correction to the wavefunction, energy levels, etc. for systems in infinite volumes. It is introduced and extensively studied by B. Sutherland [100].

- *Thermodynamic Bethe ansatz*. This technique is designed to calculate the energy levels and to investigate the thermodynamic properties of finite volume integrable systems. It is originated from the work of Yang and Yang applied for spin chains and for the Bose gas with δ interaction (the Lieb-Liniger model) [107, 108].

3.2 Heisenberg $XXX_{1/2}$ Spin Chain

We shall discuss the simplest model of quantum integrable system (no, it is not the quantum harmonic oscillator!). This model is called the *Heisenberg* $XXX_{1/2}$ *spin chain*, which was solved exactly by a young postdoc H. Bethe during a research stay in Rome in 1931. The spin chain is not just a toy model. It is a $1D$ model of magnetism which describes the dominant physical behaviour in some metals and crystals. They also have many parameters to tune and can be treated uniformly. The short chains are genuine quantum mechanical models, while long chains approximate $(1+1)D$ quantum field theories.

A spin chain is simply a $1D$ lattice with L lattice sites, and a spin-$\frac{1}{2}$ particle (for instance, an electron) is positioned on each site, labelled by the index l, with nearest neighbour spin-spin interaction. Each particle can have either spin up or down,

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.5}$$

and is therefore in a linear state $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$ generating a 2D local Hilbert space V_l. In other words, a spin can be described by an element of the local Hilbert space $V_l = V = \mathbb{C}^2$ for each l,

$$\overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_1, x_2 \in \mathbb{C}. \tag{3.6}$$

Therefore, we have a spin chain of length L (i.e. with L particles) whose states live in the full Hilbert space $\mathcal{H}$ of dimension 2^L,

$$\mathcal{H} = V^{\otimes L} = \bigotimes_L \mathbb{C}^2 = \overset{\overset{1}{\downarrow}}{\mathbb{C}^2} \otimes \ldots \otimes \overset{\overset{l}{\downarrow}}{\mathbb{C}^2} \otimes \cdots \otimes \overset{\overset{L}{\downarrow}}{\mathbb{C}^2}\,. \tag{3.7}$$

We then have a homogeneous Hamiltonian operator, $H : V^{\otimes L} \to V^{\otimes L}$, that acts on nearest neighbours

$$\hat{H} = \sum_l \hat{H}_{l,l+1}, \quad \hat{H}_{l,k} : \mathbb{V}_l \otimes \mathbb{V}_k \to \mathbb{V}_l \otimes \mathbb{V}_k, \tag{3.8}$$

where $\hat{H}_{l,k}$ is the two-site Hamiltonian. Strictly speaking, there are three Heisenberg spin Hamiltonians that describe three possible topologies for a $1D$ chain,

$$\begin{aligned} \hat{H} &= \sum_{l=1}^{L-1} \hat{H}_{l,l+1} \quad \text{(open chain)}, \\ \hat{H} &= \left(\sum_{l=1}^{L-1} \hat{H}_{l,l+1}\right) + \hat{H}_{L,1} \quad \text{(closed chain)}, \\ \hat{H} &= \sum_{l=-\infty}^{+\infty} \hat{H}_{l,l+1} \quad \text{(infinite chain)}. \end{aligned} \tag{3.9}$$

Notice that the closed chain, as depicted in Fig. 3.1, requires the periodic boundary condition $L + 1 \equiv 1$. By definition, infinite chains will have a continuous spectrum, whereas finite chains have a discrete spectrum.

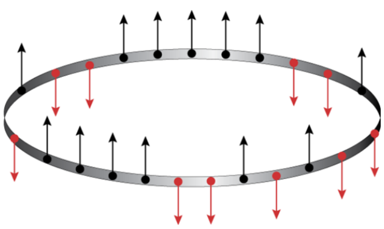


Figure 3.1 A $1D$ closed Heisenberg $XXX_{1/2}$ spin chain. A state of the spin chain can be represented as $|\psi\rangle = |\uparrow\uparrow\downarrow\uparrow\cdots\downarrow\uparrow\rangle$.

We can set up a representation $\hat{S}^\alpha, \alpha = x, y, z$, of the *global* spin algebra $\mathfrak{su}(2)$ on spin chains, which is generated by a linear combination of *local* spin operators that act nontrivially on the l^{th} space and trivially on the rest,

$$\hat{S}^\alpha = \sum_{l=1}^{L} \mathbb{1} \otimes \cdots \otimes \underbrace{\hat{S}_l^\alpha}_{l^{\text{th}} place} \otimes \cdots \otimes \mathbb{1}. \tag{3.10}$$

The local spin operators $\hat{S}_l^\alpha$ acting on each site l, are realised in terms of the standard Pauli matrices[12], $\hat{S}^\alpha = \frac{\hbar}{2}\hat{\sigma}^\alpha$, and satisfy the commutation relations

$$\left[\hat{S}_k^\alpha, \hat{S}_l^\beta\right] = i\hbar\delta_{kl}\epsilon^{\alpha\beta\gamma}\hat{S}_l^\gamma. \tag{3.11}$$

They are also subject to the periodic boundary condition $\hat{S}_l^\alpha \equiv \hat{S}_{l+L}^\alpha$.

The Hamiltonian[13] is then

$$\hat{H} = -J\sum_{l=1}^{L}\left(\hat{S}_l^\alpha \hat{S}_{l+1}^\alpha - \frac{1}{4}\mathbb{1}\right) = -J\sum_{l=1}^{L}\left(\vec{\hat{S}}_l \cdot \vec{\hat{S}}_{l+1} - \frac{1}{4}\mathbb{1}\right), \tag{3.12}$$

where J is the coupling constant, and the constant (proportional to the identity matrix) is subtracted for convenience and only shifts the energy levels by a factor. This model is usually called the *(isotropic)* $XXX_{1/2}$ *spin chain*. Notice that $J < 0$ models an anti-ferromagnetic state whereas $J > 0$ models a ferromagnetic state.

What remains is finding the spectrum of the Hamiltonian $\hat{H}$. In other words, since the Hamiltonian is a finite size matrix, we find the eigenvalues E of the matrix $\hat{H}$ by solving the characteristic equation

$$\det(\hat{H} - E\mathbb{1}) = 0. \tag{3.13}$$

One can notice that the size of the Hamiltonian is $2^L \times 2^L$. It is pretty much impossible

[12] The Pauli matrices are

$$\hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[13] A general Hamiltonian is of the form

$$\hat{H} = \sum_l \left(J_x \hat{S}_l^x \hat{S}_{l+1}^x + J_y \hat{S}_l^y \hat{S}_{l+1}^y + J_z \hat{S}_l^z \hat{S}_{l+1}^z\right).$$

This is referred to as the anisotropic XYZ spin chain. In the case $J_x = J_y$ it is called the XXZ spin chain.

to diagonalise it by brute force or numerical computation for large systems[14]. Moreover, to deal with the physical thermodynamic limit $L \to \infty$, the standard methods of linear algebra break down. We come to a conclusion that we need some alternative analytic methods to understand the spectrum.

3.3 Solving $XXX_{1/2}$ Spin Chain: Coordinate Bethe Ansatz

In what follows we choose $\hbar = 1$. H. Bethe managed to find the exact solution of the $XXX_{1/2}$ spin chain model by making a clever educated guess for the eigenstates $|\psi\rangle$ of the spectral problem, with E as the energy eigenvalues,

$$\hat{H}|\psi\rangle = E|\psi\rangle, \tag{3.14}$$

where

$$\hat{H} = J \sum_{l=1}^{L} \left(\frac{1}{4}\mathbb{1} - \hat{\vec{S}}_l \cdot \hat{\vec{S}}_{l+1} \right). \tag{3.15}$$

This guess is nowadays called the **coordinate Bethe ansatz**.

We will briefly discuss Bethe's solution of the $XXX_{1/2}$ spin chain in this section; a useful reference is [63]. The first step is to reduce the effective size of the Hamiltonian is by looking at symmetries of the system. Observe that the Hamiltonian $\hat{H}$ commutes[15] with the each spin operator $\hat{S}^\alpha$, which measures the total number of up or down spins,

$$\begin{aligned} \left[\hat{H}, \hat{S}^\alpha\right] &= -J \sum_{l,k=1}^{L} \left[\hat{S}_l^\beta \hat{S}_{l+1}^\beta, \hat{S}_k^\alpha\right] = -J \sum_{l,k=1}^{L} \left[\hat{S}_l^\beta, \hat{S}_k^\alpha\right] \hat{S}_{l+1}^\beta + \hat{S}_l^\beta \left[\hat{S}_{l+1}^\beta, \hat{S}_k^\alpha\right] \\ &= -iJ \sum_{l,k=1}^{L} \left(\delta_{lk} \epsilon^{\alpha\beta\gamma} \hat{S}_l^\beta \hat{S}_{l+1}^\gamma - \delta_{l+1,k} \epsilon^{\alpha\beta\gamma} \hat{S}_l^\gamma \hat{S}_{l+1}^\beta \right) = 0. \end{aligned} \tag{3.16}$$

One can tell that the Hamiltonian is central with respect to all $\mathfrak{su}(2)$ generators; thus, the spectrum of the model will be degenerate – all states in each $\mathfrak{su}(2)$ multiplet have the same energy. We can now restrict to subsets of a fixed number of up (or down) spins.

[14] A genuine example of a large system is a $1D$ macroscopic crystal whose number of sites of the chain L equals the number of atoms equals, say, $N_{atom} \sim 10^6$. The Hamiltonian is then a matrix of the size $2^{10^6} \times 2^{10^6}$, and equation (3.13) is an algebraic equation of degree 2^{10^6}!

[15] The identity operator commutes with all operators.

It is customary to introduce the raising and lowering operators $\hat{S}_l^{\pm} = \hat{S}_l^x \pm i\hat{S}_l^y$ where

$$\hat{S}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{S}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.17}$$

whose action on the basis vectors $|\uparrow\rangle, |\downarrow\rangle$ are

$$\begin{aligned} &\hat{S}^+|\uparrow\rangle = 0, && \hat{S}^-|\uparrow\rangle = |\downarrow\rangle, && \hat{S}^z|\uparrow\rangle = \tfrac{1}{2}|\uparrow\rangle, \\ &\hat{S}^+|\downarrow\rangle = |\uparrow\rangle, && \hat{S}^-|\downarrow\rangle = 0, && \hat{S}^z|\downarrow\rangle = -\tfrac{1}{2}|\downarrow\rangle. \end{aligned} \tag{3.18}$$

Then we can rewrite the Hamiltonian as

$$\hat{H} = \frac{JL}{4} - J\sum_{l=1}^{L} \frac{1}{2}\left(\hat{S}_l^+\hat{S}_{l+1}^- + \hat{S}_l^-\hat{S}_{l+1}^+\right) + \hat{S}_l^z\hat{S}_{l+1}^z. \tag{3.19}$$

The trick of the coordinate Bethe ansatz is to consider a reference state (vacuum state) and then flip some spins for which they behave like quasi-particles called **magnons**[16]; they can hop between sites or stay put. The spectrum can then be organised in terms of the number of flipped spins (magnons) M. For each value of M, we will first consider the case of infinite chain $L \to \infty$ and then return to the case of finite (closed) chain.

We claim that the spectrum can be found by solving the **Bethe equations** and summing the energies of the different magnons.

Claim (*Finite Closed Chain Bethe Equations*)**.** Consider a set of M algebraic equations (Bethe equations) for the M variables $u_k \in \mathbb{C}$ (Bethe roots or magnon rapidities)

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \quad \text{for } k = 1, \ldots, M. \tag{3.20}$$

For every eigenstate of $\hat{H}$ there is a solution of the above equations with $M \leq L/2$ distinct Bethe roots u_k. The energy eigenvalue of this state reads

$$E = J\sum_{k=1}^{M}\left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}}\right). \tag{3.21}$$

[16] The concept of a magnon was introduced in 1930 by Felix Bloch in order to explain the decrease of the spontaneous magnetisation in a ferromagnet [27].

Example. For $L=6, M=3$ (corresponding to a spin $\mathfrak{su}(2)$ singlet), we have

$$u_{1,2}=\pm\sqrt{-\frac{5}{12}+\frac{\sqrt{13}}{6}},\quad u_3=0,\quad E=J(5+\sqrt{13}). \tag{3.22}$$

Proof. It is instructive to re-express $\hat{H}$ acting on a spin chain as

$$\hat{H}=J\sum_{l\in\mathbb{Z}}\hat{H}_{l,l+1}=J\sum_{l\in\mathbb{Z}}(\mathbb{1}_{l,l+1}-\mathbb{P}_{l,l+1}) \tag{3.23}$$

where $\mathbb{1}$ and $\mathbb{P}$ act only on neighbouring spins at sites l and $l+1$ as the identity and permutation operators respectively,

$$\begin{aligned}\mathbb{1}_{l,l+1}|\cdots \underset{l}{\uparrow} \underset{l+1}{\downarrow} \cdots\rangle &= |\cdots \underset{l}{\uparrow} \underset{l+1}{\downarrow} \cdots\rangle,\\ \mathbb{P}_{l,l+1}|\cdots \underset{l}{\uparrow} \underset{l+1}{\downarrow} \cdots\rangle &= |\cdots \underset{l}{\downarrow} \underset{l+1}{\uparrow} \cdots\rangle.\end{aligned} \tag{3.24}$$

To be explicit[17], $\mathbb{P}_{l,l,+1}=\frac{1}{2}\mathbb{1}+2\sum_{l\in\mathbb{Z}}\hat{S}^\alpha_l\hat{S}^\alpha_{l+1}$.

Vacuum State ($M=0$). The action of $\hat{H}$ on the ferromagnetic vacuum[18] $|0\rangle=|\ldots\uparrow\uparrow\uparrow\ldots\rangle$ is trivial. Indeed,

$$\hat{H}_{l,l+1}|0\rangle=\mathbb{1}_{l,l+1}|0\rangle-\mathbb{P}_{l,l+1}|0\rangle=|0\rangle-|0\rangle=0. \tag{3.25}$$

Therefore, $\hat{H}|0\rangle=0$ and the vacuum has eigenvalue $E=0$. Here, the boundary conditions actually do not play a role and hence the same result for finite closed chain. This solves the problem for $M=0$ corresponding to the multiplet $(L/2)$.

Magnon States ($M=1$). Now flip one spin at the l site by acting with $\hat{S}^+_l$ on the vacuum,

$$|l\rangle=\hat{S}^+_l|0\rangle=|\ldots\uparrow\uparrow\underset{l}{\downarrow}\uparrow\uparrow\cdots\rangle. \tag{3.26}$$

This state enumerated by l forms a closed sector under the Hamiltonian due to conservation of the z-component of spin, $\hat{S}^z$. Hence it is not an eigenstate of $\hat{H}$. Instead we

[17] We have modified the coefficient of identity operator in (3.15) to $\frac{1}{2}$, which is largely irrelevant to the proof because it merely induces an overall shift of all energies. Our choice sets the energy of the vacuum state to zero; otherwise, it would be $E_0=-\frac{JL}{4}$ in (3.25).

[18] We pick the most numerous particle state as the reference vacuum. If the number of the spin-up particles $n_\uparrow$ is bigger than or equal to the number of the spin-down particles $n_\downarrow=L-n_\uparrow$, i.e. $n_\uparrow\geq n_\downarrow$, we choose $|\uparrow\rangle\otimes\ldots\otimes|\uparrow\rangle=|\uparrow\ldots\uparrow\rangle$ as the reference vacuum. It is also known as the *pseudo-vacuum* in some texts.

can try the momentum eigenstate[19], which is simply a Fourier transform of a position eigenstate, called a *magnon state*,

$$|p\rangle = \sum_{l\in\mathbb{Z}} \psi_p(l)|l\rangle \quad \psi_p(l) = \exp(ipl). \tag{3.27}$$

It can be viewed as a particle excitation of the vacuum state. Here the notion of particle is an object which carries an individual momentum p. Indeed we can check that

$$\begin{aligned}\hat{H}|l\rangle &= J\sum_{k\in\mathbb{Z}}\Big[\overbrace{\delta_{k,l}\left(|l\rangle - |l+1\rangle\right)}^{\hat{H}_{k,k+1}|l\rangle} + \overbrace{\delta_{k,l}\left(|l\rangle - |l-1\rangle\right)}^{\hat{H}_{k-1,k}|l\rangle}\Big] \\ &= J[2\,|l\rangle - |l+1\rangle - |l-1\rangle]\end{aligned} \tag{3.28}$$

and so,

$$\begin{aligned}\hat{H}|p\rangle &= \sum_{l\in\mathbb{Z}} \exp\left(\mathrm{i}pl\right)\hat{H}\,|l\rangle \\ &= J\sum_{l\in\mathbb{Z}} \exp\left(\mathrm{i}pl\right)\left[2\,|l\rangle - |l+1\rangle\,| - |l-1\rangle\right] \\ &= J\sum_{l\in\mathbb{Z}} \exp\left(\mathrm{i}pl\right)\left[2 - \exp(\mathrm{i}p) - \exp(-\mathrm{i}p)\right]|l\rangle \\ &= 2J(1-\cos p)|p\rangle \\ &= 4J\sin^2\left(\frac{p}{2}\right)|p\rangle\end{aligned} \tag{3.29}$$

confirming that $|p\rangle$ is an eigenstate with the *magnon dispersion relation*,

$$E(p) = 4J\sin^2\left(\frac{1}{2}p\right). \tag{3.30}$$

For an infinite chain, p is a continuous parameter. For a finite closed chain, the momentum is quantised by the periodic boundary conditions $\exp[ip(l+L)] = \exp(ipl) \Rightarrow \exp(ipL) = 1$ to

$$p = \frac{2\pi n}{L} \quad (\bmod 2\pi), \qquad \text{where } n = 0, \ldots, L-1. \tag{3.31}$$

This solves the problem for $M = 1$ corresponding to the multiplets $(L/2 - 1)$.

Scattering States ($M = 2$). We continue with states with two spin flips. When the two magnons are well-separated ($l_2 > l_1 + 2$), they behave as two single magnons,

$$|l_1, l_2\rangle = |\ldots\uparrow\underset{l_1}{\downarrow}\uparrow\ldots\uparrow\underset{l_2}{\downarrow}\uparrow\ldots\rangle, \text{ with } l_1 < l_2. \tag{3.32}$$

[19]The momentum carried by a magnon is known as *pseudo-momentum* in some texts.

The natural ansatz for wavefunction is of the form

$$|\psi\rangle = |p_1, p_2\rangle = \sum_{l_1 < l_2} \psi_{p_1,p_2}(l_1, l_2) |l_1, l_2\rangle, \text{ with } p_1 > p_2, \tag{3.33}$$

where

$$\psi_{p_1,p_2}(l_1, l_2) = \exp[i(p_1 l_1 + p_2 l_2)] + \mathcal{S}(p_1, p_2) \exp[i(p_1 l_2 + p_2 l_1)]. \tag{3.34}$$

The first term on the right hand side of the above equation corresponds to a partial wave describing two incident magnons propagating,

$$|\ldots \uparrow\uparrow \underset{l_1}{\overset{\rightarrow p_1}{\downarrow}} \uparrow\uparrow \ldots \uparrow\uparrow \underset{l_2}{\overset{\leftarrow p_2}{\downarrow}} \uparrow\uparrow \ldots\rangle,$$

while the second term corresponds to partial wave describing two transmitted[20] magnons propagating,

$$|\ldots \uparrow\uparrow \underset{l_1}{\overset{\leftarrow p_2}{\downarrow}} \uparrow\uparrow \ldots \uparrow\uparrow \underset{l_2}{\overset{\rightarrow p_1}{\downarrow}} \uparrow\uparrow \ldots\rangle.$$

As usual in scattering, the coefficients of the incident and transmitted waves differ by a phase factor $\mathcal{S}(p_1, p_2)$ known as the *S-matrix*. It is given by

$$\mathcal{S}(p_1, p_2) = \frac{u(p_1) - u(p_2) + i}{u(p_1) - u(p_2) - i}, \tag{3.35}$$

where $u(p) = \cot(p/2)/2$ is known as the *magnon rapidity* [8]. Reality of the energy eigenvalues requires,

$$\mathcal{S}(p_1, p_2) = \mathcal{S}^{-1}(p_2, p_1). \tag{3.36}$$

A spacetime picture of the scattering process is shown in Fig. 3.2.

We can immediately show that the wavefunction $\psi_{p_1,p_2}(l_1, l_2)$ given in (3.34) is an eigenfunction of $\hat{H}$ with eigenvalue,

$$E(p_1, p_2) = E(p_1) + E(p_2) = 4J \sin^2\left(\frac{p_1}{2}\right) + 4J \sin^2\left(\frac{p_2}{2}\right). \tag{3.37}$$

This reflects the fact that the magnons propagate freely on the spin chain, and their energies simply add up. There are two exceptions to the scattering states, which we will not consider here, i.e. when the two magnons are next to each other ($l_2 = l_1 + 1$) and bound states of the two magnons. For a full discussion of these cases, see [43].

[20] The processes of transmission and reflection are indistinguishable for identical particles in one spatial dimension. Therefore we do not need to include a separate contribution to account for reflection.

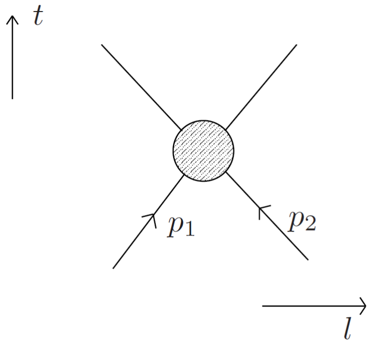


Figure 3.2 Two magnon scattering process.

Now let us consider the finite chain of length L. Observe that (3.34) implies that

$$\psi_{p_1,p_2}(l_1+L,l_2) = \exp(ip_1L)\exp[i(p_1l_1+p_2l_2)] + \exp(ip_2L)\,\mathcal{S}(p_1,p_2)\exp[i(p_1l_2+p_2l_1)] \tag{3.38}$$

and if the periodic boundary conditions

$$\psi_{p_1,p_2}(l_1+L,l_2) = \psi_{p_1,p_2}(l_1,l_2+L) = \psi_{p_1,p_2}(l_1,l_2) \tag{3.39}$$

hold, then

$$\exp(\mathrm{i}p_1L) = \mathcal{S}(p_1,p_2)\,, \quad \exp(\mathrm{i}p_2L) = \mathcal{S}(p_2,p_1)\,. \tag{3.40}$$

The quantisation of the two magnon momenta is therefore given by the two-body S-matrix.

Multi-magnons ($M>2$). In the same fashion we expand a general state in terms of position eigenstates for M magnons,

$$|\psi\rangle = \sum_{l_1<l_2<\ldots<l_M} \psi_{p_1,p_2,\ldots,p_M}(l_1,l_2,\ldots,l_M)\,|l_1,l_2\cdots l_M\rangle\,. \tag{3.41}$$

A general scattering state for M magnons with momenta $p_1,p_2,\ldots,p_M$ is given by the coordinate Bethe ansatz,

$$\psi_{p_1,p_2,\ldots,p_M}(l_1,l_2,\ldots,l_M) = \sum_{\sigma\in\Pi_M} \mathcal{S}^{(M)}_\sigma(p_1,\ldots,p_M)\exp[i\left(p_1l_{\sigma(1)}+p_2l_{\sigma(2)}+\cdots+p_Ml_{\sigma(M)}\right)] \tag{3.42}$$

The wavefunction is a summation over all possible partial waves up to permutation σ of the integers $\{1,2,\ldots,M\}$. Each partial wave is weighted by a corresponding phase determined by an *M-body S-matrix*, $\mathcal{S}^{(M)}_\sigma(p_1,\ldots,p_M)$.

A key simplification we can do here is to perform exact factorisation[21] of the multi-magnon scattering amplitude as a product of two-magnon scattering amplitudes. For instance, in the scattering of three magnons with momenta $p_1 > p_2 > p_3$, the three-body S-matrix $\mathcal{S}^{(3)}_\sigma(p_1, p_2, p_3)$ with $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ can be factorised exactly as

$$\mathcal{S}^{(3)}_\sigma(p_1, p_2, p_3) = \mathcal{S}(p_2, p_3)\, \mathcal{S}(p_1, p_3)\, \mathcal{S}(p_1, p_2)\,, \tag{3.43}$$

where the two-body S-matrix is again given by

$$\mathcal{S}(p_1, p_2) = \frac{u(p_1) - u(p_2) + i}{u(p_1) - u(p_2) - i} = \frac{u_1 - u_2 + i}{u_1 - u_2 - i}, \tag{3.44}$$

as shown in Fig. 3.3.

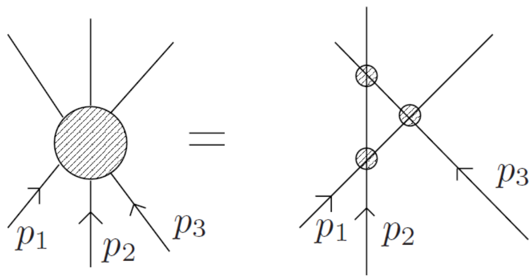


Figure 3.3 Factorisation of three magnon scattering.

To complete the proof, we consider again the finite chain. The periodic boundary conditions give rise to quantisation of M-body momenta, known as the *Bethe ansatz equation* (BAE),

$$\exp(\mathrm{i} p_k L) = \prod_{\substack{j=1 \\ j\neq k}}^{M} \mathcal{S}(p_k, p_j)\,. \tag{3.45}$$

The interpretation is simple. We take j^{th} magnon and move it around the circle (spin chain). Once it encounters another magnon, they scatter with each other and it picks up a scattering phase. Written out in terms of the rapidity, it simply becomes

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \qquad \text{for } k = 1, \ldots, M. \tag{3.46}$$

[21] We used the fact that any permutation $\sigma \in \Pi_M$ can be factorised as a composition of transpositions of adjacent elements. Each transposition corresponds to a factor of the two body S-matrix $\mathcal{S}$.

The total energy eigenvalue is just the sum of individual magnon energies,

$$E\left(p_1, \ldots, p_M\right) = \sum_{k=1}^{M} 4J \sin^2\left(\frac{p_k}{2}\right). \tag{3.47}$$

In terms of rapidity, the total energy eigenvalue becomes

$$E = J \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}} = J \sum_{k=1}^{M} \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}}\right). \tag{3.48}$$

□

The coordinate Bethe ansatz marked the discovery of quantum integrability in one dimension, which interestingly happened while the general formalism of non-relativistic quantum mechanics was still being developed. Moreover, at the beginning of the 21st century, an reincarnation of this method occurred in the context of integrability in AdS/CFT. This method proved to be useful in solving problems in higher dimensional theories, i.e. in supersymmetric gauge field theories and string theory.

3.4 Algebraic Bethe Ansatz

In order to have a more transparent view of integrability of the $XXX_{1/2}$ spin chain, we can approach the problem using a variation of Bethe ansatz, called the **algebraic Bethe ansatz**. A basic tool for this approach is the **Lax operator**,

$$L_{l,a}(u): \quad V_l \otimes V_a \to V_l \otimes V_a \tag{3.49}$$

and it is explicitly given by

$$L_{l,a}(u) = u\mathbb{1}_l \otimes \mathbb{1}_a + i\hat{S}_l^\alpha \otimes \hat{\sigma}^\alpha = \begin{pmatrix} u + i\hat{S}_l^z & i\hat{S}_l^- \\ i\hat{S}_l^+ & u - i\hat{S}_l^z \end{pmatrix}_a, \quad l = 1, 2, \ldots, L, \tag{3.50}$$

where $u \in \mathbb{C}$ is the spectral parameter. Here, $\mathbb{1}_l$ and $\hat{S}_l^\alpha$ act on the local Hilbert space $V_l = \mathbb{C}^2$, while $\mathbb{1}_a$ and the Pauli matrices $\hat{\sigma}^\alpha$ act on another Hilbert space $V_a = \mathbb{C}^2$, called *auxiliary space*. We can also write it in terms of a permutation operator, $\mathbb{P} = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} + \hat{S}^\alpha \otimes \hat{\sigma}^\alpha$,

$$L_{l,a}(u) = \left(u - \frac{i}{2}\right)\mathbb{1}_{l,a} + i\mathbb{P}_{l,a}. \tag{3.51}$$

Consider taking two Lax operators, $L_{l,a}(u_1)$ and $L_{l,b}(u_2)$, acting on the same local Hilbert space but on two different auxiliary spaces. The products of these two operators are defined in a triple tensor product $V_l \otimes V_a \otimes V_b$ and satisfy the *fundamental commutation relation*

$$R_{a,b}(u_1 - u_2)\, L_{l,a}(u_1)\, L_{l,b}(u_2) = L_{l,b}(u_2)\, L_{l,a}(u_1)\, R_{a,b}(u_1 - u_2)\,. \tag{3.52}$$

The intertwining operator is called **quantum R-matrix**, which is explicitly given by

$$R_{a,b} = u\mathbb{1}_{a,b} + i\mathbb{P}_{a,b}. \tag{3.53}$$

It is convenient to suppress the index of the local Hilbert space and rewrite the fundamental commutation relation as

$$R_{ab}(u_1 - u_2)\, L_a(u_1)\, L_b(u_2) = L_b(u_2)\, L_a(u_1)\, R_{ab}(u_1 - u_2) \tag{3.54}$$

What the R-matrix does is that it interchanges the position of the operators (can be thought of as being 2×2 matrices) L_1 and L_2. Essentially, a triple product

$$\begin{aligned} L_1 L_2 L_3 &= R_{12}^{-1} L_2 L_1 R_{12} L_3 = R_{12}^{-1} L_2 L_1 L_3 R_{12} \\ &= R_{12}^{-1} R_{13}^{-1} L_2 L_3 L_1 R_{13} R_{12} = R_{12}^{-1} R_{13}^{-1} R_{23}^{-1} L_3 L_2 L_1 R_{23} R_{13} R_{12} \end{aligned} \tag{3.55}$$

brings the product $L_1 L_2 L_3$ to the form $L_3 L_2 L_1$. However, the same effect can be reached by changing the order of permutations

$$\begin{aligned} L_1 L_2 L_3 &= R_{23}^{-1} L_1 L_3 L_2 R_{23} = R_{23}^{-1} R_{13}^{-1} L_3 L_1 L_2 R_{13} R_{12} \\ &= R_{12}^{-1} R_{13}^{-1} L_2 L_3 L_1 R_{13} R_{12} = R_{23}^{-1} R_{13}^{-1} R_{12}^{-1} L_3 L_2 L_1 R_{12} R_{13} R_{23}. \end{aligned} \tag{3.56}$$

Therefore, a condition is imposed on the R-matrix,

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \tag{3.57}$$

This is the **quantum Yang-Baxter equation**.

Using the R-matrix, we can construct the **monodromy operator**,

$$T_0(u) = L_L(u) L_{L-1}(u) \ldots L_2(u) L_1(u), \quad \text{with} \quad L_n \equiv R_{0n}. \tag{3.58}$$

We shall obtain a tower of conserved quantities by constructing the **transfer matrix**

$\mathcal{T}(u)$ (cf. Section 2.3),

$$\mathcal{T}(u) = \mathrm{Tr}_0\, T_0(u), \tag{3.59}$$

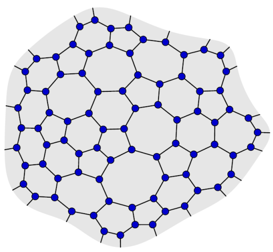
and hence we check that the spin chain is integrable [8, 43].

Part II

Integrability of $\mathcal{N} = 4$ SYM Theory

Chapter 4

$\mathcal{N} = 4$ Super Yang-Mills Theory



A planar Feynman diagram.

4.1 $\mathcal{N} = 4$ SYM – A Cousin of QCD

On our journey of understanding the behaviour of elementary particles, Yang-Mills (YM) theory is at the core of the Standard Model of particle physics. YM theory is a nonabelian gauge theory based on the $SU(N)$ group. This theory plays an important role in the theoretical formulation of electroweak and strong interactions, which are based on $U(1) \times SU(2)$ and $SU(3)$ groups, respectively.

The $\mathcal{N} = 4$ super Yang-Mills (SYM) theory[22] in four-dimensional Minkowski spacetime is a highly symmetric version of YM theory. $\mathcal{N} = 4$ SYM has the maximal amount of supersymmetry without spin-two fields (gravitons). It is also an exact conformal field theory (CFT) in which the physics remains the same under any rescaling of energies or length scales. It is much like the theory of Quantum Chromodynamics (QCD) that

[22]The $\mathcal{N}$ here is the number of copies of the supersymmetry algebra, i.e. not the rank of the gauge group, N.

describes the strong interaction, albeit QCD is not conformal[23]. Having a conformal invariance implies that the theory is not confining; hence there are no physical particles (for instance, mesons and hadrons) whose mass spectrum we might wish to compute. Nevertheless, it still serves as a useful tool to understand several aspects of strong interaction, such as the high energy gluon scattering that can be learned by studying gauge boson amplitudes in $\mathcal{N} = 4$ SYM.

Conformal invariance of YM theory and QCD with massless quarks are broken by an anomaly in the quantum theory. The symptom of this breaking is the nonvanishing of the β function, $\beta(g) = \mu\partial g/\partial\mu < 0$, which leads to a running coupling $g^2(\mu)$ with dependence on the renormalisation group (RG) scale μ (see Fig. 4.1).

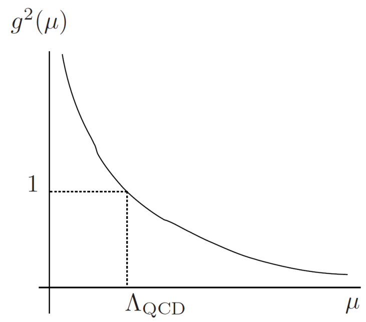


Figure 4.1 The running coupling in QCD. The cutoff Λ_{QCD} breaks the conformal invariance of QCD.

In contrast, $\mathcal{N} = 4$ SYM stays conformally invariant even at the quantum level! Its β-function is zero to all orders in perturbation theory [97], as was first conjectured in [55] when studying open string loop amplitudes which reduce to ten-dimensional SYM in the infinite string tension limit. Arguments for a nonperturbative vanishing of the $\beta-$function are given in [93]. As the dimensionless coupling g^2 does not run, the $SO(4, 2)$ conformal invariance of the classical $\mathcal{N} = 4$ SYM theory is unbroken. The large amount of symmetry in this theory thus leads to an underlying integrability, making many physical quantities analytically calculable and many consequences of the AdS/CFT conjecture computationally verifiable.

[23]QCD is asymptotically free; hence it is close to being conformal at high energies. It also has a running coupling constant, and thus there is a natural energy scale of 200MeV at the crossover point from weak to strong coupling. This point is roughly where confinement sets in and is responsible for the proton mass.

4.2 Action and Symmetries

$\mathcal{N}=4$ SYM was first considered by Brink, Scherk and Schwarz [29], who explicitly constructed its Lagrangian density by dimensionally reducing SYM from 10 to 4 dimensions. Let us consider the ten-dimensional $\mathcal{N}=1$ gauge theory

$$S=\int d^{10}x\,\mathrm{Tr}\left(-\frac{1}{4}\mathcal{F}_{MN}\mathcal{F}^{MN}+\frac{i}{2}\overline{\Psi}\Gamma^{M}\mathcal{D}_{M}\Psi\right),\quad M,N=1,\ldots,10, \tag{4.1}$$

where Γ^M are the gamma matrices in ten dimensions. Here, we use the Lorentzian metric with signature $(-++\ldots++)$. The covariant derivative is defined as

$$\mathcal{D}_M\equiv\partial_M+ig_{\mathrm{YM}}\left[\mathcal{A}_M,\cdot\right], \tag{4.2}$$

where the dot refers to the field being acted on by the covariant derivative, and g_{YM} is the Yang-Mills coupling constant. Ψ is a sixteen-component Majorana-Weyl spinor of $SO(9,1)$. One could perform dimensional reduction to this theory and anticipate that six of ten components of the gauge field become scalars, while the remaining four are still gauge fields. The sixteen-dimensional spinor decomposes into four copies of left and right-handed Weyl spinors in four dimensions.

$$\begin{array}{llcll}
\mathcal{A}_M, & M=1,\ldots,4 & \rightarrow & \mathcal{A}_\mu, & \mu=0,\ldots,3\\
\mathcal{A}_M, & M=5,\ldots,10 & \rightarrow & \Phi_i, & i=1,\ldots,6\\
\Psi_A, & A=1,\ldots,16 & \rightarrow & \Psi_{\alpha a},\overline{\Psi}^{a}_{\dot\alpha}, & a=1,\ldots,4,\quad \alpha,\dot\alpha=1,2.
\end{array} \tag{4.3}$$

We define the covariant derivative in four dimensions,

$$\mathcal{D}_\mu\equiv\partial_\mu+ig_{\mathrm{YM}}\left[\mathcal{A}_\mu,\cdot\right]. \tag{4.4}$$

The real fields Φ and Ψ with the covariant derivative $\mathcal{D}_\mu$ (regarded as one of the fundamental fields) form a supermultiplet $\mathcal{W}=\left(\mathcal{D}_\mu,\Psi_{\alpha a},\overline{\Psi}^{a}_{\dot\alpha},\Phi_i\right)$. All components in $\mathcal{W}$ transform in the adjoint representation of the $SU(N)$ gauge group and they can be represented by traceless, hermitian $N\times N$ matrices. Under a gauge transformation $U(x)\in SU(N)$ the fields transform canonically as

$$\mathcal{W}\mapsto U\mathcal{W}U^{-1},\quad \mathcal{A}_\mu\mapsto U\mathcal{A}_\mu U^{-1}+\frac{i}{g_{\mathrm{YM}}}\partial_\mu UU^{-1}. \tag{4.5}$$

The gauge field $\mathcal{A}_\mu$ transforms differently from the other fields to compensate for the noncovariant transformation of the partial derivative within $\mathcal{D}$. Since the covariant derivative $\mathcal{D}$ is not truly a field, we construct a field from the gauge connection alone, i.e. the field strength $\mathcal{F}$, with the associated Bianchi identity,

$$\mathcal{F}_{\mu\nu} = ig_{\mathrm{YM}}^{-1}\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + ig_{\mathrm{YM}}\left[\mathcal{A}_\mu, \mathcal{A}_\nu\right], \quad \mathcal{D}_{[\rho}\mathcal{F}_{\mu\nu]} = 0. \tag{4.6}$$

The action for the four-dimensional $\mathcal{N} = 4$ SYM theory[24] is then

$$S = \frac{2}{g_{\mathrm{YM}}^2}\int d^4x\, \mathcal{L}_{\mathrm{YM}} \tag{4.7}$$

with the Lagrangian density

$$\begin{aligned} \mathcal{L}_{\mathrm{YM}} = \mathrm{Tr}\Bigg(&-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{2}\mathcal{D}_\mu\Phi_i\mathcal{D}^\mu\Phi^i + \frac{1}{4}\left[\Phi_i, \Phi_j\right]\left[\Phi^i, \Phi^j\right] \\ &- \overline{\Psi}^a_{\dot\alpha}\sigma_\mu^{\dot\alpha\beta}\mathcal{D}^\mu\Psi_{\beta a} + \frac{i}{2}\Psi_{\alpha a}\sigma_i^{ab}\epsilon^{\alpha\beta}\left[\Phi^i, \Psi_{\beta b}\right] + \frac{i}{2}\overline{\Psi}^a_{\dot\alpha}\sigma^i_{ab}\epsilon^{\dot\alpha\dot\beta}\left[\Phi_i, \overline{\Psi}^b_{\dot\beta}\right]\Bigg), \end{aligned} \tag{4.8}$$

where σ^μ and σ^i are the chiral projections of the gamma matrices in four and six dimensions respectively. The symbols ϵ are the $\mathfrak{su}(2)$ totally antisymmetric tensors. There is a global $SO(6) \simeq SU(4)$ symmetry, called an *R-symmetry* (or *flavour symmetry*), with the scalars Φ_i transforming in the second rank complex self-dual (vector) $\mathbf{6}$ representation of $\mathfrak{su}(4)$; the spinors $\Psi_{\alpha a}$ in the (fundamental) $\mathbf{4}$ and $\overline{\Psi}^a_{\dot\alpha}$ in the (antifundamental) $\overline{\mathbf{4}}$; the gauge field $\mathcal{A}_\mu$ is a singlet.

In addition to being Poincaré invariant, the action (4.7) is scale invariant. Specifically, the $\mathcal{N} = 4$ SYM theory is invariant under *dilatations* or scale transformations,

$$D : x_\mu \to \lambda x_\mu, \tag{4.9}$$

which act on the classical fields X of the $\mathcal{N} = 4$ theory as,

$$D : X\left(x_\mu\right) \to \lambda^{\Delta_0} X\left(\lambda x_\mu\right), \tag{4.10}$$

where $\Delta_0 = [X]$ is the mass dimension of the field X. The various fields in the theory

[24] In principle, $\mathcal{N} = 4$ SYM theory also has a vacuum (instanton) angle θ which combines with g_{YM} into a complex coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}$. Through the AdS/CFT duality, the angle θ equals the expectation value of the axion field in the spectrum of the dual type IIB superstring [34]. Nevertheless, this parameter will be neglected as it plays no role in the following discussion.

have *classical dimensions*

$$\Delta_0 = \begin{cases} [\mathcal{A}_\mu] = [\Phi_i] = 1, \\ [\Psi_{\alpha a}] = \left[\overline{\Psi}^a_{\dot\alpha}\right] = \frac{3}{2}. \end{cases} \tag{4.11}$$

Since $\mathcal{N}=4$ theory is four-dimensional (i.e. higher than two-dimensional theory), it is also invariant under the special conformal transformations,

$$K_\mu : x_\mu \to \frac{x_\mu + a_\mu x^2}{1 + 2x^v a_v + a^2 x^2}. \tag{4.12}$$

The *bosonic* part of the theory has a global symmetry group, $SO(4,2)\times SO(6)$. The former factor is the four-dimensional conformal group, $SO(4,2)\simeq SU(2,2)$, which has 15 generators spanning its algebra $\mathfrak{so}(4,2)\simeq\mathfrak{su}(2,2)$: ten generators belong to the Poincaré algebra, i.e. four translation generators, P_μ, and six generators of the $\mathfrak{so}(1,3)\simeq\mathfrak{su}(2)\times\mathfrak{su}(2)$ Lorentz transformations, $M_{\mu\nu}$, and one dilatation generator, D, and four generators of special conformal transformations, K_μ. These generators satisfy the commutation relations

$$\begin{aligned} &[D, P_\mu] = -iP_\mu, \quad [D, M_{\mu\nu}] = 0, \quad [D, K_\mu] = +iK_\mu, \\ &[M_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho}P_\nu - \eta_{\rho\nu}P_\mu\right), \quad [M_{\mu\nu}, K_\rho] = -i\left(\eta_{\mu\rho}K_\nu - \eta_{\rho\nu}K_\mu\right), \\ &[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} \pm \text{permutations}, \quad [P_\mu, K_\nu] = 2i\left(M_{\mu\nu} - \eta_{\mu\nu}D\right), \end{aligned} \tag{4.13}$$

with all other commutators vanishing and $\eta_{\mu\nu}$ is the Minkowski metric. The latter factor is again the R-symmetry, which is an internal symmetry that rotates the fields into one another.

Taking into account the supersymmetry transformations, the generators are *fermionic* and are called **supercharges**. There are 16 distinct supercharges $Q_{\alpha a}, \widetilde{Q}^a_{\dot\alpha}$ for the $\mathcal{N}=4$ supersymmetry, which satisfy the commutation and anticommutation relations

$$\begin{aligned} &\left\{Q_{\alpha a}, \widetilde{Q}^b_{\dot\alpha}\right\} = \gamma^\mu_{\alpha\dot\alpha}\delta^b_a P_\mu, \quad \{Q_{\alpha a}, Q_{\alpha b}\} = \left\{\widetilde{Q}^a_{\dot\alpha}, \widetilde{Q}^b_{\dot\alpha}\right\} = 0, \\ &[P_\mu, Q_{\alpha a}] = \left[P_\mu, \widetilde{Q}^b_{\dot\alpha}\right] = 0, \\ &[M^{\mu\nu}, Q_{\alpha a}] = i\gamma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q_{\gamma a}, \quad \left[M^{\mu\nu}, \widetilde{Q}^a_{\dot\alpha}\right] = i\gamma^{\mu\nu}_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma}\widetilde{Q}^a_{\dot\gamma}, \end{aligned} \tag{4.14}$$

where $\gamma^{\mu\nu}_{\alpha\beta} = \gamma^{[\mu}_{\alpha\dot\alpha}\gamma^{\nu]}_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}$. Dimension counting within the algebra shows that $Q_{\alpha a}$ and $\widetilde{Q}^a_{\dot\alpha}$ have dimension $\frac{1}{2}$ and so their commutators with D is

$$[D, Q_{\alpha a}] = -\frac{i}{2}Q_{\alpha a}, \quad \left[D, \widetilde{Q}^a_{\dot\alpha}\right] = -\frac{i}{2}\widetilde{Q}^a_{\dot\alpha}. \tag{4.15}$$

By commuting the special conformal generators K_μ with $Q_{\alpha a}$ and $\widetilde{Q}^a_{\dot\alpha}$,

$$[K^\mu, Q_{\alpha a}] = \gamma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\widetilde{S}_{\dot\beta a}, \quad \left[K^\mu, \widetilde{Q}^a_{\dot\alpha}\right] = \gamma^\mu_{\alpha\dot\alpha}\epsilon^{\alpha\beta}S^a_\beta, \tag{4.16}$$

we generate a new set of operators S^a_α and $\widetilde{S}_{\dot\alpha a}$, which have dimension $-\frac{1}{2}$. They are known as the special conformal supercharges, or the **superconformal charges**. Their R-charge representations are reversed from the supercharges and combine with the regular supercharges to give 32 supercharges in total. The superconformal charges have anticommutation relations that mirror the anticommutation relations of the supercharges,

$$\begin{aligned} \left\{S^a_\alpha, \widetilde{S}_{\dot\alpha b}\right\} &= \gamma^\mu_{\alpha\dot\alpha}\delta^a_b K_\mu, \\ [K_\mu, S^a_\alpha] &= \left[K_\mu, \widetilde{S}_{\dot\alpha a}\right] = 0. \end{aligned} \tag{4.17}$$

Anticommutation relations between the supercharges and the superconformal charges complete the algebra,

$$\begin{aligned} \left\{Q_{\alpha a}, S^b_\beta\right\} &= -i\epsilon_{\alpha\beta}\sigma^{IJ}_a R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta^b_a M_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta^b_a D, \\ \left\{\widetilde{Q}^a_{\dot\alpha}, \widetilde{S}_{\dot\beta b}\right\} &= +i\epsilon_{\dot\alpha\dot\beta}\sigma^{IJa}R_{IJ} + \gamma^{\mu\nu}_{\dot\alpha\dot\beta}\delta^a_b M_{\mu\nu} - \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\delta^a_b D, \\ \left\{Q_{\alpha a}, \widetilde{S}_{\dot\beta b}\right\} &= \left\{\widetilde{Q}^a_{\dot\alpha}, S^b_\beta\right\} = 0. \end{aligned} \tag{4.18}$$

On the right-hand side of (4.18) we obtain in addition to the Lorentz and dilatation generators the $SO(6) \simeq SU(4)$ R-symmetry generators R_{IJ}, where $I, J = 1 \ldots 6$. The supercharges transform under the two-spinor representations of $SO(6)$, while all generators of the conformal algebra are singlets and commute with R_{IJ}. Altogether the 30 bosonic generators and 32 fermionic generators span the $\mathcal{N} = 4$ superconformal group denoted $\widetilde{PSU}(2,2|4)$ with its algebra $\mathfrak{psu}(2,2|4)$ (see Appendix for details).

4.3 The Large N Expansion

In QCD, perturbation theory in the gauge coupling $g^2(\mu)$ is only useful in the UV regime such as deep inelastic scattering. Infrared (IR) physics such as quark confinement and chiral symmetry breaking is nonperturbative in $g^2(\mu)$. In 1973, a genuine idea was put forward by 't Hooft [101] to replace the $SU(3)$ colour gauge group of QCD by $SU(N)$ and take the *'t Hooft limit*, $N \to \infty$ with the *'t Hooft coupling*, $\lambda = g^2_{\text{YM}}N$, held fixed[25]. Corrections to this limit are considered as a power series in $1/N$. Such an expansion

[25] N is simply the number of colours, N_C, in QCD.

scheme is sensible as the leading order of large N theory still exhibits confinement and chiral symmetry breaking.

We will now study the large N expansion at the level of Feynman diagrams. With the 't Hooft coupling, the $\mathcal{N} = 4$ SYM action (4.7) looks like

$$S = \frac{2N}{\lambda} \int d^4x \, \mathcal{L}_{\mathrm{YM}}. \tag{4.19}$$

Recall that each adjoint field is an $N \times N$ matrix, $(\mathcal{W}_\mu)^i_j\,, i, j = 1, \ldots, N$. We will focus on the gauge field $\mathcal{A}_\mu$. The propagator has the index structure

$$\left\langle \mathcal{A}^i_{\mu j}(x) \mathcal{A}^k_{\nu l}(y) \right\rangle = \Delta_{\mu\nu}(x-y) \left(\delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l \right), \tag{4.20}$$

where $\Delta_{\mu\nu}(x)$ is the usual bosonic propagator for a single gauge field. The $1/N$ term is clearly suppressed by $1/N$, we do not lose anything by dropping this term at leading order in $1/N$. We then have

$$\left\langle \mathcal{A}^i_{\mu j}(x) \mathcal{A}^k_{\nu l}(y) \right\rangle = \Delta_{\mu\nu}(x-y) \delta^i_l \delta^k_j \tag{4.21}$$

and only a single leading order diagram remains.

This can be depicted using 't Hooft's *double line notation* (see Fig. 4.2) which relies on the decomposition of the adjoint representation of $SU(N)$,

$$\mathbf{adj} \equiv \mathbf{N} \otimes \overline{\mathbf{N}} - \mathbf{1}, \tag{4.22}$$

where $\mathbf{N}$ and $\overline{\mathbf{N}}$ are the fundamental and anti-fundamental representation of $SU(N)$, respectively. The upper and lower lines with opposite arrows are associated to complex conjugate representations. The propagator scales as λ/N, as can be read off from the action (4.19).

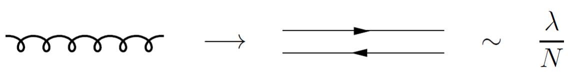
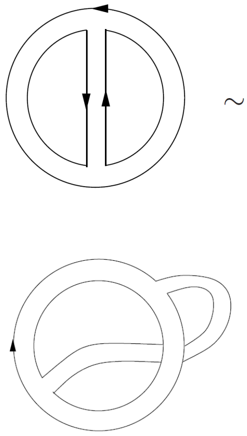


Figure 4.2 Double-line notation for a gluon propagator instead of the usual curly line notation.

The Feynman rules for the gauge fields in $\mathcal{N} = 4$ SYM (similarly, gluons in QCD) are such that each edge corresponds to a propagator weighted with $g^2_{\mathrm{YM}} = \lambda/N$, and each vertex is weighted with $1/g^2_{\mathrm{YM}} = N/\lambda$. The general scaling of a Feynman diagram amplitude is

$$\text{diagram} \sim \left(\frac{\lambda}{N}\right)^{\#\text{ propagators}} \left(\frac{N}{\lambda}\right)^{\#\text{ vertices}} N^{\#\text{ index contractions}}, \tag{4.23}$$

where # denotes number, and the index contractions come from the loops in the diagram. One can have planar[26] and nonplanar diagrams (for instance, see the gluon vacuum bubble in Fig. 4.3). At large N limit, nonplanar diagram in Fig. 4.3b is suppressed by $1/N^2$ relative to the planar diagram.



$$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N^3 \sim \lambda N^2$$

(a)

$$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N \sim \lambda$$

(b)

Figure 4.3 (a) Planar vacuum bubble, (b) nonplanar vacuum bubble.

't Hooft showed that the "fat" Feynman diagrams can be classified by their topology [101]. Each diagram corresponds to the triangulation of a Riemann surface, as shown in Fig. 4.4. A Feynman diagram with V vertices, E edges (or propagators), and F faces (or loops) scales like

$$N^{V-E+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V} = N^{2-2g}\lambda^{E-V}, \tag{4.24}$$

where $\chi = V - E + F$ is the Euler character[27] of the Riemann surface and g is the genus (i.e. hole or handle) of the surface.

In the limit $N \to \infty$, the leading order given by planar diagrams dominate; hence the large N theory will be simpler than real world QCD (SU(3)) (see Fig. 4.5). Moreover, the $1/N$ expansion classified by the topology of Riemann surfaces matches the perturbation

[26] Planar diagrams are those that can be drawn on a plane without lines intersecting.

[27] The Euler character is $\chi = 2 - 2g$ if the Feynman diagram (or correspondingly the Riemann surface) is closed and orientable.

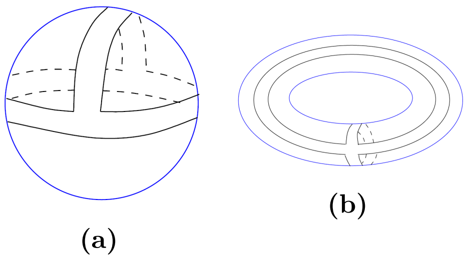


Figure 4.4 (a) The planar vacuum bubble in Fig. 4.3a can be drawn on a two-sphere ($\chi = 2$), whereas (b) the nonplanar diagram in Fig. 4.3b must be drawn on a higher-genus surface such as the torus ($\chi = 0$).

theory of closed string sigma model with a string coupling $g_s \sim 1/N$. Therefore, a hope emerges through the seminal work of 't Hooft that QCD, or at least large N strongly coupled SYM theory, can be described in terms of a weakly coupled string theory.

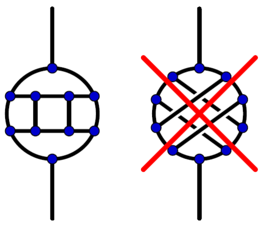


Figure 4.5 Gauge theory at large-N limit where $N = \infty$, $g_{\text{YM}} = 0$ and λ remains. Only planar Feynman diagrams (left) without crossing propagators remains, leading to drastic combinatorial simplification.

4.4 Local Operators

When SYM theory is quantised, the big symmetry group $\widetilde{PSU}(2,2|4)$, including the classical conformal symmetry is unbroken by quantum corrections. This significant constraint on the quantised theory provides us with a powerful tool. In this section we will sketch how the action of the dilatation operator[28] D, one of the generator of $\widetilde{PSU}(2,2|4)$ group, leads to the existence of operators of minimal scaling dimension, called primary operators. Notice that while the generators of the Poincaré subgroup of $\widetilde{PSU}(2,2|4)$ do not get quantum corrections, the dilatation operator D does:

$$D = D_0 + \delta D(g_{\text{YM}}), \tag{4.25}$$

[28]The hats on operators will be dropped from now on for convenience.

where D_0 is the classical operator and δD is the anomalous dilatation operator which depends on the YM coupling g_{YM}.

Now let $\mathcal{O}(x)$ be a *local operator* in the field theory with scaling dimension Δ. Under a scaling $x \to \lambda x$, $\mathcal{O}(x)$ scales as $\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)$; D is the generator of these scalings, $\mathcal{O}(x) \to \lambda^{-iD}\mathcal{O}(x)\lambda^{iD}$, and its action on $\mathcal{O}(x)$ is

$$[D, \mathcal{O}(x)] = i\left(-\Delta + x\frac{\partial}{\partial x}\right)\mathcal{O}(x). \tag{4.26}$$

The dimension Δ is $\Delta_0 + \gamma(g_{\mathrm{YM}}^2)$ with Δ_0 the classical dimension corresponding to the classical operator D_0 and $\gamma(g_{\mathrm{YM}}^2)$ the anomalous dimension arising from quantum corrections corresponding to δD [35] (see Chapter 5 for discussion of anomalous dimension computation).

Primary operator. Let us act D on the commutator $[K_\mu, \mathcal{O}(0)]$ and find, using the Jacobi identity, that

$$\begin{aligned}[D, [K_\mu, \mathcal{O}(0)]] &= [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)]] \\ &= i\,[K_\mu, \mathcal{O}(0)] - i\Delta\,[K_\mu, \mathcal{O}(0)]\,.\end{aligned} \tag{4.27}$$

We see that the special conformal generator K_μ acts as a *lowering operator* which creates a new local operator, $[K_\mu, \mathcal{O}(0)]$ of one lower dimension $\Delta - 1$. Aside from the identity operator, unitary quantum field theories must have local operators whose dimensions are positive (in fact a nonconstant operator must have $\Delta \geqslant 2$). Therefore, as we keep lowering the dimension by acting with K_μ, we must eventually reach a lower bound where

$$\left[K_\mu, \widetilde{\mathcal{O}}(0)\right] = 0 \quad \forall K_\mu. \tag{4.28}$$

Such an operator $\widetilde{\mathcal{O}}(x)$ is known as a *primary operator*[29]. Starting with $\widetilde{\mathcal{O}}(x)$, we can build higher-dimensional operators, known as *descendants* of $\widetilde{\mathcal{O}}(x)$, by acting on it with the *raising operator*, P_μ, an arbitrary number of times, where $[P_\mu, \mathcal{O}(x)] = -i\partial_\mu\mathcal{O}(x)$. The primary operator and its descendants make up an irreducible representation of $\widetilde{PSU}(2,2|4)$, with the primary as the highest weight of the representation. $\widetilde{PSU}(2,2|4)$ is noncompact, so the representation is infinite dimensional.

Chiral primary operator. A similar analysis can be done with the fermionic generators, i.e. the supercharges and superconformal charges. The supercharges $Q_{\alpha a}$ and $\widetilde{Q}^a_{\dot\alpha}$

[29] The primary condition (4.28) is defined at $x = 0$ where the spacetime position is a fixed point of the dilatation. If the local operator were at a different spacetime point then it would commute with a different combination of the conformal generators.

can make new operators with 1/2 higher dimension. These operators along with the original operators make up a supermultiplet. Since the supercharges are fermionic, the maximal number of operators in a supermultiplet is $2^{16/2}=2^8=256$. The number of operators in the supermultiplet may be reduced by annihilating some operators with the superconformal charges S^a_α and $\widetilde{S}_{\dot{\alpha}a}$. It is clear that the operator must be a primary, otherwise the anticommutation relation in (4.17) would not lead to (4.28). Hence we have

$$\left[S^a_\alpha, \widetilde{\mathcal{O}}(0)\right] = \left[\widetilde{S}_{\dot{\alpha}a}, \widetilde{\mathcal{O}}(0)\right] = 0 \quad \forall \alpha, \dot{\alpha}, a. \tag{4.29}$$

We now put a further restriction that

$$[Q_{\alpha a}, \mathcal{O}(0)] = 0 \qquad \text{for some } \alpha, a. \tag{4.30}$$

Such operators are known as *chiral primaries*. Consistency now requires that

$$\begin{aligned}\left[\left\{Q_{\alpha a}, S^b_\beta\right\}, \widetilde{\mathcal{O}}(0)\right] &= \left\{Q_{\alpha a}, \left[S^b_\beta, \mathcal{O}(0)\right]\right\} + \left\{S^b_\beta, [Q_{\alpha a}, \mathcal{O}(0)]\right\} = 0 \\ &= \left[-i\epsilon_{\alpha\beta}\sigma^{IJ}_a R_{IJ} + \gamma^{\mu\nu}_{\alpha\beta}\delta^b_a M_{\mu\nu} - \frac{1}{2}\epsilon_{\alpha\beta}\delta^b_a D, \mathcal{O}(0)\right].\end{aligned} \tag{4.31}$$

If $\widetilde{\mathcal{O}}(0)$ is a scalar field, $M_{\mu\nu}$ commutes with it and we get a relation between the R-charge of $\widetilde{\mathcal{O}}(0)$ and its dimension Δ,

$$\sigma^{IJ}{}_a{}^b \left[R_{IJ}, \widetilde{\mathcal{O}}(0)\right] = \Delta \delta_a{}^b \widetilde{\mathcal{O}}(0). \tag{4.32}$$

To find what the relations are, we first observe that $SO(6) \simeq SU(4)$ is a rank 3 group, and thus has three commuting generators in its Cartan subalgebra, which we choose to be R_{12}, R_{34} and R_{56}. The corresponding charges for these can be written as $[J_1, J_2, J_3]$. The $\sigma^{IJ}{}_a{}^b$ are the generators in the $SU(4)$ fundamental representation, with

$$\sigma^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma^{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma^{56} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.33}$$

Therefore, a primary operator with R-charges $[J, 0, 0]$ is consistently annihilated by $Q_{\alpha 1}$ and $Q_{\alpha 2}$ if $\Delta = J$. Here, $J = J_1 = J_2 = J_3$ since $J_i, i = 1,2,3$ states are all in the same $\widetilde{PSU}(2,2|4)$ representation. Inspection of (4.14) shows that they are also annihilated by $\widetilde{Q}^3_{\dot{\alpha}}$ and $\widetilde{Q}^4_{\dot{\alpha}}$. Such operators commute with (or are annihilated by) half the supercharges and are known as chiral primaries (or $\frac{1}{2}$-BPS operators)[30]. Therefore, the number of operators in the supermultiplet reduces to $2^{8/2} = 2^4 = 16$.

The great advantage of chiral primaries is that their dimensions are *protected*. This means that their dimensions do not vary with the coupling constant[31], and hence commute with half the supercharges no matter what the coupling. If they did not commute then there would have to be extra operators at each level. Therefore, the number of operators in the supermultiplet is a finite integer that stays fixed. The R-charges are also integers; thus the dimensions must stay fixed, i.e. $\Delta = \Delta_0$.

$\mathcal{N} = 4$ SYM theory is characterised by its spectrum of *local gauge invariant operators* $\mathcal{O}(x)$, which transform in unitary irreducible representations of $SO(4,2) \times SO(6)$ – the bosonic subgroup of $\widetilde{PSU}(2,2|4)$. The Dynkin labels for a state of the operator consist of the roots (or Cartan eigenvalues) of its algebra, i.e. a sextuplet of charges, $[\Delta, S_1, S_2, J_1, J_2, J_3]$. Here Δ is the scaling dimension, $S_r\,(r = 1,2)$ are the spins, i.e. charges of the $SO(1,3)$ Lorentz group $\subset SO(4,2)$. The $J_i\,(i = 1,2,3)$ are the three commuting R-charges which correspond to $U(1)_R \subset SO(6) \simeq SU(4)$. Apart from Δ, these charges correspond to generators of compact subgroups of $SO(4,2) \times SO(6)$ and are therefore quantised in integer units.

Appendix: Superconformal Algebra

Symmetry is a principle that organises objects with similar properties and often leads to structural constraints on correlation functions. The use of superconformal symmetry has helped to verify the AdS/CFT duality at various stages. An immediate check for the AdS/CFT conjecture is that the two models have coincident global symmetries: $\mathcal{N} = 4$ superconformal symmetry on the one hand and the isometries of the $AdS_5 \times S^5$ superspace on the other. They are both given by the Lie supergroup $\widetilde{PSU}(2,2|4)$ with its algebra $\mathfrak{psu}(2,2|4)$. Here, $\widetilde{PSU}(2,2|4)$ is the non-trivial universal cover of $PSU(2,2|4)$ where its abelian subgroup is noncompact. This appendix demonstrates the purely algebraic aspects of $\mathfrak{psu}(2,2|4)$ (see [13, 46] for details). A formal definition of the superalgebra $\mathfrak{psu}(2,2|4)$ is given as follows.

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[30] BPS stands for Bogomol'nyi–Prasad–Sommerfield. Chiral primary operators can be classified into $\frac{1}{2}$ − BPS, $\frac{1}{4}$ − BPS, and $\frac{1}{8}$ − BPS [36, 91]. The $\frac{1}{2}$ − BPS live in $[J,0,0]$ representations of the R-symmetry group $SU(4)$; their scaling dimension is $\Delta = J$ and are annihilated by eight out of the sixteen Poincaré supercharges. Similarly, $\frac{1}{4}$ − BPS primaries are in the $[J_1, J_2, J_1]$ representations, are annihilated by four supercharges, and have protected scaling dimension of $\Delta = 2J_1 + J_2$. The $\frac{1}{8}$ − BPS primaries belong to $[J_1, J_2, J_1 + 2J_3]$, are killed by only two supercharges, and their $\Delta = 2J_1 + J_2 + 2J_3$.

[31] In general, an operator depends on the YM coupling constant g_{YM}.

Definition. The algebra $\mathfrak{psu}(2,2|4)$ is a $(30|32)$-dimensional real Lie superalgebra.

To understand how we can obtain this definition, let us begin with the general superalgebra. As we have seen in section 4.2, a superalgebra contains bosonic generators B_μ and fermionic generators F_α. An element of a superalgebra is a linear combination,

$$X = X^\mu B_\mu + \theta^\alpha F_\alpha, \tag{4.34}$$

where X^μ and θ^α are Grassmann-even and Grassmann-odd variables, respectively. The generators B_μ and F_α can be represented in finite dimension as ordinary numeric matrices. An element of the Lie superalgebra X then forms a supermatrix, which encodes the linear transformations of $\mathbb{R}^{(n|m)}$ or $\mathbb{C}^{(n|m)}$, a space of vectors whose first n components are Grassmann-even numbers and the last m ones are Grassmann-odd:

$$\left(\begin{array}{c|c} E & O \\ \hline O' & E' \end{array}\right)\left(\begin{array}{c} b \\ \hline f \end{array}\right). \tag{4.35}$$

Let us consider complex $(4|4)$-dimensional square supermatrices

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right). \tag{4.36}$$

Each block A, B, C, D is a 4×4 matrix of complex numbers. The blocks A, D are considered even and B, C odd. These supermatrices span a Lie superalgebra of $\mathfrak{gl}(4|4,\mathbb{C})$, which is a $(32|32)$-dimensional vector space. This superalgebra is not simple; thus having nontrivial ideals. A conventional way to get a simple algebra is to impose the traceless condition, but here a deviation from the usual Lie algebras occurs. The Lie superalgebra $\mathfrak{gl}(4|4,\mathbb{C})$ satisfies a graded Lie bracket $[\cdot,\cdot\}$, which is the graded commutator of supermatrices

$$[X,Y\} = XY - (-1)^{XY}YX = \left(\begin{array}{c|c} AE+BG-EA+FC & AF+BH-EB-FD \\ \hline CE+DG-GA-HC & CF+DH+GB-HD \end{array}\right), \tag{4.37}$$

where Y is the analogue of X in (4.36) with blocks E, F, G, H. This differs from a usual commutator through the signs for the odd-odd products FC and GB. Therefore, the trace of a graded Lie bracket is not zero.

The nonvanishing trace can be fixed by using the *supertrace* $\operatorname{Str} X := \operatorname{Tr} A - \operatorname{Tr} D$, i.e. a subtraction between traces of matrices is used instead of the usual addition.

The supertrace is zero for graded commutator (4.37), $\operatorname{Str}[X,Y\} = 0$. Demanding the *supertraceless* condition ($\operatorname{Str} X = 0$) thus removes a degree of freedom from $\mathfrak{gl}(4|4,\mathbb{C})$, and restricts it to the subalgebra $\mathfrak{sl}(4|4,\mathbb{C})$. Furthermore, the identity supermatrix $\mathbb{1}$ commutes with all other matrices, $[\mathbb{1}, Y\} = 0$; thus it generates the centre and can be *projected* out from $\mathfrak{sl}(4|4,\mathbb{C})$ yielding a $(30|32)$-dimensional complex Lie superalgebra $\mathfrak{psl}(4|4,\mathbb{C})$.

To further restrict to the *real form* $\mathfrak{psu}(2,2|4)$ we impose a *hermiticity* condition on the supermatrices

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} HA^\dagger H^{-1} & -iHC^\dagger \\ \hline -iB^\dagger H^{-1} & D^\dagger \end{array}\right), \tag{4.38}$$

where H is a hermitian matrix of signature $(2,2)$. There are two natural choices for H. First, H is a matrix of 2×2 diagonal blocks ('$+$'/'$-$' denotes the 2×2 positive/negative identity matrix),

$$H = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}, \quad X = \left(\begin{array}{cc|c} M_1 & iN & -iQ_1 \\ i\overline{N} & M_2 & +iQ_2 \\ \hline \overline{Q}_1 & \overline{Q}_2 & R \end{array}\right), \quad X' = \left(\begin{array}{c|c|c} M_1 & -iQ_1 & iN \\ \hline \overline{Q}_1 & R & \overline{Q}_2 \\ \hline i\overline{N} & +iQ_2 & M_2 \end{array}\right). \tag{4.39}$$

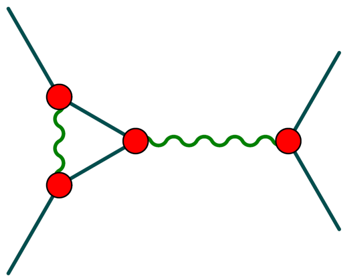
Here the hermitian blocks M_1 and M_2 generate the maximal compact subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) = \mathfrak{so}(4) \oplus \mathfrak{so}(2)$ of $\mathfrak{su}(2,2) = \mathfrak{so}(4,2)$. It is often convenient to reorder the $2,2|4$ rows and columns, and move one of the 2's past the 4. X' is thus the supermatrix X reordered in $2|4|2$-block form. This choice of H is useful in the context of the AdS_5 spacetime and for unitary representations. The second choice is an off-diagonal H.

$$H = \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}, \quad X = \left(\begin{array}{cc|c} L & P & -iQ \\ K & \overline{L} & -i\overline{S} \\ \hline S & \overline{Q} & R \end{array}\right), \quad X' = \left(\begin{array}{c|c|c} L & -iQ & P \\ \hline S & R & \overline{Q} \\ \hline K & -i\overline{S} & \overline{L} \end{array}\right). \tag{4.40}$$

Now the hermitian conjugate blocks L, $\overline{L}$ in X generate the Lorentz and scaling transformations in $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{gl}(1) = \mathfrak{so}(3,1) \oplus \mathfrak{so}(1,1)$. This choice is obviously adapted to four-dimensional Minkowski space.

Chapter 5

$\mathcal{N} = 4$ SYM One-loop Integrability



An $\mathcal{N} = 4$ SYM one-loop diagram.

5.1 Anomalous Dimensions

The natural observables of the $\mathcal{N} = 4$ SYM theory are *correlation functions of local gauge-invariant operators*. The local gauge-invariant operators are made by taking traces of the product of the adjoint fields $\mathcal{W} = \left(\mathcal{D}_\mu(x), \Psi_{\alpha a}(x), \overline{\Psi}^a_{\dot{\alpha}}(x), \Phi_i(x)\right)$, evaluated at the same spacetime point x. For instance, consider a *single trace operator*,

$$\mathcal{O}_1 = \mathrm{Tr}\left(\Psi_{\alpha a}\mathcal{D}_\mu\Phi^i\mathcal{D}_v\overline{\Psi}^a_{\dot{\alpha}}\right), \tag{5.1}$$

which has a well-defined classical scaling dimension Δ_0 (the sum of the mass dimensions of each field). For the operator $\mathcal{O}_1$, we find $\Delta_0 = 6$. There also exist *multitrace operators* such as

$$\mathcal{O}_2 = \mathrm{Tr}\left(\Phi^i\Phi^j\right)\mathrm{Tr}\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right), \tag{5.2}$$

which do not survive in the 't Hooft limit ($N \to \infty$). The dimensions of multitrace operators will just be the sums of the dimensions of the single traces that make up the operator.

The correlation functions of these operators $\mathcal{O}_i$ at different spacetime points are defined by introducing sources $J_i(x)$ for each operator in the generating functional,

$$\begin{aligned} Z\left[\{J_i\}\right] &= \int \mathrm{d}A\,\mathrm{d}\Psi\,\mathrm{d}\overline{\Psi}\,\mathrm{d}\Phi\,\exp\left(\frac{i}{\hbar}\int \mathrm{d}^4x\mathcal{L}_{\mathrm{YM}} + \sum_{i=1}^{M} J_i(x)\mathcal{O}_i(x)\right), \\ \langle \mathcal{O}_1\left(x_1\right)\mathcal{O}_2\left(x_2\right)\cdots\mathcal{O}_M\left(x_M\right)\rangle &= \frac{\delta^M}{\delta J_1\left(x_1\right)\delta J_2\left(x_2\right)\cdots\delta J_M\left(x_M\right)} Z\left[\{J_i\}\right]. \end{aligned} \tag{5.3}$$

Consider the two-point correlation function of a local operator $\mathcal{O}(x)$ with itself. It is constrained by the Poincaré symmetry to be

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle = f(x-y) \tag{5.4}$$

with $f(x)$ being an arbitrary scalar function; a further conformal symmetry constraint fixes its form to

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta}}, \tag{5.5}$$

where the dimension $\Delta = \Delta_0 + \gamma$, with Δ_0 being the (classical) bare dimension and γ being the anomalous dimension arising from quantum corrections.

For local operators made up only of scalars with no covariant derivatives, $\mathcal{O}(x) = \mathrm{Tr}\,\Phi^L$, since all scalars have bare dimension 1, the bare dimension is $\Delta_0 = L$, the number of scalar fields inside the trace. At the tree-level, the two-point function[32] of the local operator is equivalent to the L scalar propagators up to a constant c,

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle_{\mathrm{tree}} = \frac{c}{|x-y|^{2\Delta_0}} \approx \left(\frac{1}{|x-y|^2}\right)^L = \frac{c}{|x-y|^{2L}}. \tag{5.6}$$

If g_{YM} is small, then $\gamma \ll \Delta_0$. In this case one can approximate the two-point function as

$$\langle \mathcal{O}(x)\overline{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta_0}}\left(1 - \gamma\ln(\Lambda^2|x-y|^2)\right), \tag{5.7}$$

[32] The tree-level three-point function is of the form

$$\langle \mathcal{O}_A\left(x_1\right)\mathcal{O}_B\left(x_2\right)\mathcal{O}_C\left(x_3\right)\rangle_{\mathrm{tree}} = \frac{c_{ABC}}{|x_{12}|^{\Delta_{0A}+\Delta_{0B}-\Delta_{0C}}\,|x_{23}|^{\Delta_{0B}+\Delta_{0C}-\Delta_{0A}}\,|x_{31}|^{\Delta_{0C}+\Delta_{0A}-\Delta_{0B}}}$$

up to a constant c_{ABC}. The higher-order correlation functions can be expressed as a product of two-point and three-point functions since the theory is conformally invariant.

where Λ is cutoff scale. The leading contribution to this correlator is thus called the tree-level contribution.

The computation of anomalous dimensions simplifies tremendously in the 't Hooft limit. However, this computation is complicated by the problem of operator mixing (see next section). Miraculously, the mixing can often be restricted to operators within certain "closed" sectors.

5.2 Mixing Matrix and the Relation to Spin Chains

Recall that the matter content of $\mathcal{N}=4$ SYM includes six adjoint scalars Φ_i. We can express them as three complex fields, $Z=\frac{1}{\sqrt{2}}(\Phi_1+i\Phi_2)$, $W=\frac{1}{\sqrt{2}}(\Phi_3+i\Phi_4)$, $X=\frac{1}{\sqrt{2}}(\Phi_5+i\Phi_6)$, along with their conjugates. All scalars in 4 dimensions have (classical) bare dimension $\Delta_0=1$ and are of course spinless, and thus the R-charges for Z,W and X are, respectively, $[1,0,0,1,0,0]$, $[1,0,0,0,1,0]$ and $[1,0,0,0,0,1]$. Their conjugates, $\overline{Z}$, $\overline{W}$ and $\overline{X}$ have reversed R-charges.

As we let $N\to\infty$, we will see that the planar diagrams dominate, much like the gluon vacuum bubble in Section 4.3. This can be shown by contraction of chiral primaries. Let us consider the rescaled chiral primary,

$$\Psi_L=\frac{\left(4\pi^2\right)^{L/2}}{\sqrt{L}N^{L/2}}\operatorname{Tr}Z^L=\frac{\left(4\pi^2\right)^{L/2}}{\sqrt{L}N^{L/2}}Z^A{}_BZ^B{}_C\ldots Z^{\cdots}{}_A\quad A,B,C=1,\ldots,N, \tag{5.8}$$

where the colour indices have been written out explicitly. The prefactors are normalisation factors. At tree level, the correlator of a Z field and its conjugate $\overline{Z}$ is

$$\left\langle Z^A{}_B(x)\overline{Z}^C{}_D(y)\right\rangle_{\text{tree}}=\frac{\delta^A{}_D\delta_B{}^C}{4\pi^2|x-y|^2}, \tag{5.9}$$

where we have ignored the fact that $Z^A{}_A=0$ which is justifiable in the large N limit. The leading contribution to a contraction between Ψ_L and its conjugate $\overline{\Psi}_L$ comes from contracting the individual fields in order, as shown in Fig. 5.1a and 5.1b. The contribution of all such ordered contractions is

$$\left\langle\Psi_L(x)\overline{\Psi}_L(y)\right\rangle_{\text{ordered}}=\frac{LN^L}{\left(\sqrt{L}N^{L/2}\right)^2|x-y|^{2L}}=\frac{1}{|x-y|^{2L}}. \tag{5.10}$$

where the L powers of N come from contractions of neighboring fields, $N=\delta^{A'}{}_A\delta^A{}_{A'}$. We also pick up a factor of L from the L ways of contracting the fields in the plane.

Fig. 5.1c is an example of a nonplanar diagram due to the fact that at least one connecting line must be lifted out of the plane to avoid such crossings. Comparing to (a), we would see that there are two less factors of N in (c),

$$\begin{aligned} &\text{(a)} \quad \ldots \delta^{A'}{}_{A}\delta^{A}{}_{A'}\delta^{B'}{}_{B}\delta^{B}{}_{B'}\delta^{C'}{}_{C}\delta^{C}{}_{C'} \cdots = \ldots N^3 \ldots , \\ &\text{(c)} \quad \ldots \delta^{A'}{}_{A}\delta^{A}{}_{B'}\delta^{C'}{}_{B}\delta^{B}{}_{A'}\delta^{B'}{}_{C}\delta^{C}{}_{C'} \cdots = \ldots N \ldots , \end{aligned} \tag{5.11}$$

where the dots represent contractions that are the same in both cases. Hence, the nonplanar diagram in (c) (in fact, each nonplanar diagram) is suppressed by a factor of $1/N^2$, as we would have expected. We can thus ignore the contribution of all nonplanar diagrams in the 't Hooft limit[33].

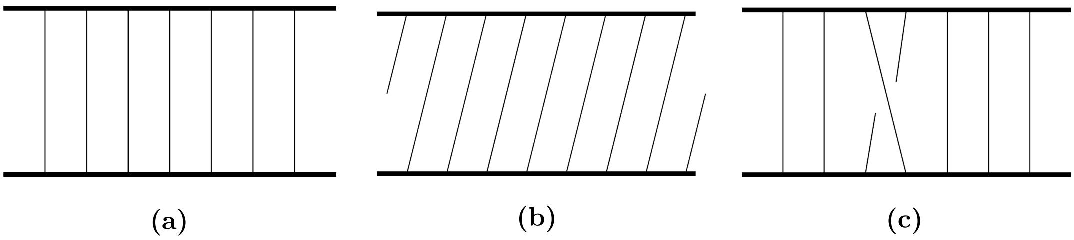


Figure 5.1 Contractions of fields. The horizontal lines represent the operators and the ordered vertical lines the contractions between the two operators of the individual fields inside the trace. (a) and (b) are planar, while (c) is nonplanar.

Without going into much details [76], the tree-level correlator in (5.10) can be generalised to any scalar operator of the form

$$\mathcal{O}_{I_1,I_2,\ldots,I_L}(x) = \frac{(4\pi^2)^{L/2}}{\sqrt{C_{I_1,I_2,\ldots,I_L}}N^{L/2}} \operatorname{Tr}\left(\Phi_{I_1}(x)\Phi_{I_2}(x)\ldots\Phi_{I_L}(x)\right), \tag{5.12}$$

where $C_{I_1,I_2,\ldots,I_L}$ is a symmetry factor (which is n if the flavour indices $I_i, i = 1,\ldots,L$ are

[33]Note that this analysis is valid so long as $L \ll N$. Otherwise, if L were on the order of N, then the suppression coming from the $1/N$ factors is swamped by the huge number of nonplanar diagrams compared to the number of planar diagrams. (There are $L!$ total tree level diagrams of which only L are planar.)

invariant when shifting by L/n). Adding the one-loop correlator[34], we find the expression,

$$\begin{aligned}&\left\langle \mathcal{O}_{I_1,I_2,\ldots,I_L}(x)\overline{\mathcal{O}}^{J_1,J_2,\ldots,J_L}(y)\right\rangle \\ &= \frac{1}{|x-y|^{2L}}\left(1-\frac{\lambda}{16\pi^2}\ln\left(\Lambda^2|x-y|^2\right)\sum_{l=1}^{L}(1-C-2P_{l,l+1}+K_{l,l+1})\right)\delta_{I_1}^{J_1}\delta_{I_2}^{J_2}\ldots\delta_{I_L}^{J_L} \\ &\quad + \text{cycles},\end{aligned} \tag{5.13}$$

where "cycles" refers to the $L-1$ cyclic shifts of the J_i indices and λ is the 't Hooft coupling. There is a sum over l because the interaction between any of the L pairs of neighboring fields exists in the planar diagram in Fig. 5.2a. The constant C comes from other planar diagrams in Fig. 5.2c, 5.2d, and 5.2e.

We have defined two important operators, namely the exchange operator $P_{l,l+1}$ and the trace operator $K_{l,l+1}$. The operator $P_{l,l+1}$ exchanges the flavor indices of the l and the $l+1$ sites inside the trace, whereas $K_{l,l+1}$ contracts the flavor indices of neighboring fields. Their actions on the δ-functions in (5.13) are, respectively,

$$\begin{aligned}P_{l,l+1}\delta_{I_1}^{J_1}\ldots\delta_{I_l}^{J_l}\delta_{I_{l+1}}^{J_{l+1}}\ldots\delta_{I_L}^{J_L} &= \delta_{I_1}^{J_1}\ldots\delta_{I_l}^{J_{l+1}}\delta_{I_{l+1}}^{J_l}\ldots\delta_{I_L}^{J_L}, \\ K_{l,l+1}\delta_{I_1}^{J_1}\ldots\delta_{I_l}^{J_l}\delta_{I_{l+1}}^{J_{l+1}}\ldots\delta_{I_L}^{J_L} &= \delta_{I_1}^{J_1}\ldots\delta_{I_lI_{l+1}}\delta^{J_lJ_{l+1}}\ldots\delta_{I_L}^{J_L}.\end{aligned} \tag{5.14}$$

Therefore, we have operator mixing at the one-loop level due to $P_{l,l+1}$ and $K_{l,l+1}$.

When we compare this result to (5.7), we see that because of the operator mixing the anomalous dimension γ should be replaced with an operator Γ where

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}(1-C-2P_{l,l+1}+K_{l,l+1}). \tag{5.15}$$

The possible one-loop anomalous dimensions are then found by diagonalising Γ, which is indeed a *mixing matrix*. However, with Bethe ansatz at hand, we can skip the tedious steps to compute and diagonalise the mixing matrix Γ.

One could be concerned that, since we have operator mixing, scalar field operators will mix with nonscalar field operators. It turns out that this does not happen at the one-loop level although it generally can. To see this, consider the complete dilatation

[34] The one-loop contribution to the two-point correlator comes mainly from the quartic scalar vertex given in the bosonic part of the $\mathcal{N}=4$ action (4.7),

$$S = \frac{1}{2g_{\text{YM}}^2}\int d^4x\,\text{Tr}\left\{-\mathcal{F}^2 - 2\mathcal{D}_\mu\Phi_I\mathcal{D}^\mu\Phi^I + \sum_{I<J}[\Phi_I,\Phi_J]^2\right\}.$$

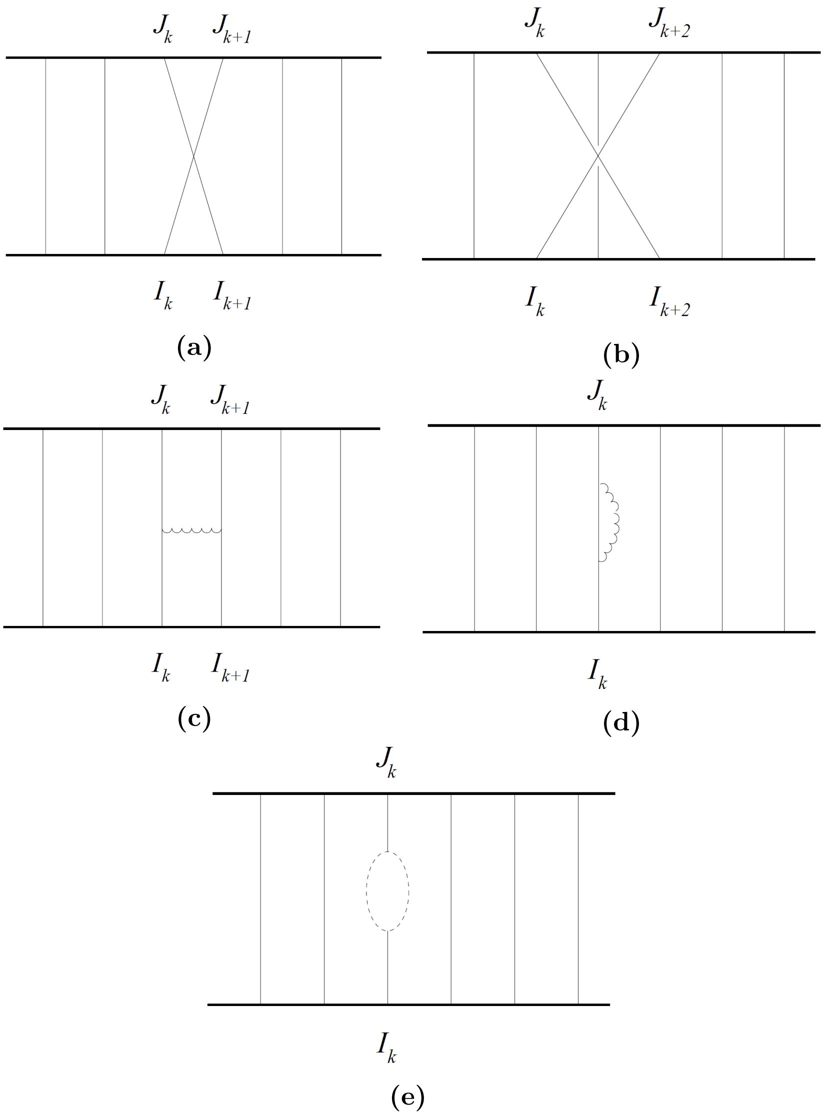


Figure 5.2 Quartic scalar interaction contributes to the one-loop correlator, where (a) two neighbouring fields are connected and (b) two nonneighbouring fields are connected. Notice that the interaction has added a loop to the diagrams, and case (b) is nonplanar. Other one-loop planar diagrams (c), (d), and (e) do not affect the flavour structures. (c) A gluon exchange between neighbouring scalars. Since the gluon carries no R-charge, the flavour indices are unchanged. (d) Scalar self-energy from a gluon; (e) scalar self-energy from a fermion loop. R-charge conservation and the fact that only one scalar line is involved means that (d) and (e) leave the flavour indices unchanged.

operator, which can be expressed as an expansion in λ of the form

$$D = \sum_{n=0}^{\infty} \lambda^n D^{(2n)}, \tag{5.16}$$

$D^{(0)}$ gives the bare dimension of the operator, while $D^{(2)}$ is the one-loop anomalous dimension mixing matrix Γ in (5.15). The dilation operator commutes with the Lorentz generators and the R-symmetry generators. This is true for all values of λ; hence all $D^{(2n)}$ commute with these generators. Moreover, each of the $D^{(2n)}$ commutes with $D^{(0)}$, which can be established by power counting in the graphs. Therefore, mixing only occurs between operators with the same Lorentz charges, R-charges, and bare dimensions. In other words, operator mixing preserves the total charges of the $\widetilde{PSU}(2,2|4)$ symmetry group. An overview of the correspondences between the observables in $\mathcal{N}=4$ SYM at 't Hooft limit and integrable spin chain is provided in Table 5.1.

Table 5.1 Correspondences between $\mathcal{N}=4$ SYM at 't Hooft limit and integrable spin chain.

$\mathcal{N}=4$ **SYM at** $N\to\infty$	**Integrable spin chain**
Single trace operator	Closed spin chain
Field operator	Spin at a site
Anomalous dilatation operator $\delta D(g^2_{\mathrm{YM}})$	Hamiltonian H
Anomalous dimension γ	Energy eigenvalue E

5.3 Closed Sector Bethe Ansatz at One Loop

The mixing matrix Γ at one loop was first computed in [22]. Minahan and Zarembo [77] noticed that, for the $SO(6)$ closed sector[35] spanned by the 6 scalars, Γ can be treated as the Hamiltonian on the integrable $SO(6)$ spin chain (Fig. 5.3), and is solved by Reshetikhin [86, 87]. This has been the starting point for all the integrability machinery in AdS/CFT.

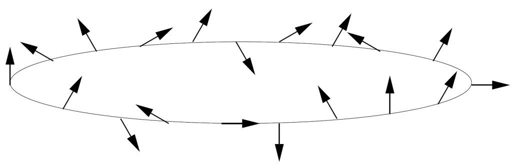


Figure 5.3 A spin chain with $SO(6)$ vector sites.

One way to find the $SO(6)$ sector mixing matrix $\Gamma_{SO(6)}$ is to use a particular eigenstate – the chiral primary Ψ_L in (5.8). Ψ_L is symmetric under the exchange of any field; thus $P_{l,l+1}\Psi_L = \Psi_L$ for any l. Furthermore, Ψ_L has only Z fields and not $\overline{Z}$ fields, hence

[35] It should be noted that the $SO(6)$ sector is not closed at higher loop order and it mixes with sectors containing fermions.

$K_{l,l+1}\Psi_L = 0$. This generalises to any chiral primary, which is in the L^{th} symmetric traceless representation of $SO(6)$. Therefore,

$$\Gamma_{SO(6)}\Psi_L = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}(1-C-2)\Psi_L. \tag{5.17}$$

However, the dimension of Ψ_L is protected, meaning that its anomalous dimension is zero. We then find that $C=-1$ and Γ becomes

$$\Gamma_{SO(6)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1-P_{l,l+1}+\frac{1}{2}K_{l,l+1}\right). \tag{5.18}$$

Instead of following Reshetikhin's method [87], one can also obtain the solution using the *nested Bethe ansatz*, proposed by Kulish and Reshetikhin [33]. This method extend the algebraic Bethe ansatz to algebras of higher rank. It is called *nested* because at each step of the procedure the rank of the algebra is reduced. The Hamiltonian corresponding to $\Gamma_{SO(6)}$,

$$H^{(2)}_{SO(6)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1-P_{l,l+1}+\frac{1}{2}K_{l,l+1}\right), \tag{5.19}$$

is diagonalised in terms of the rapidities satisfying the nested Bethe ansatz equations,

$$\begin{aligned}
1 &= \prod_{l\neq k}^{M_1}\frac{u_{1,k}-u_{1,l}+i}{u_{1,k}-u_{1,l}-i}\prod_{l\neq k}^{M_2}\frac{u_{1,k}-u_{2,l}-i/2}{u_{1,k}-u_{2,l}+i/2}\\
\left(\frac{u_{2,k}+i/2}{u_{2,k}-i/2}\right)^L &= \prod_{l\neq k}^{M_2}\frac{u_{2,k}-u_{2,l}+i}{u_{2,l}-u_{2,l}-i}\prod_{l\neq k}^{M_1}\frac{u_{2,k}-u_{1,l}-i/2}{u_{2,k}-u_{1,l}+i/2}\prod_{l\neq k}^{M_3}\frac{u_{2,k}-u_{3,l}-i/2}{u_{2,k}-u_{3,l}+i/2}\\
1 &= \prod_{l\neq k}^{M_3}\frac{u_{3,k}-u_{3,l}+i}{u_{3,k}-u_{3,l}-i}\prod_{l\neq k}^{M_2}\frac{u_{3,k}-u_{2,l}-i/2}{u_{3,k}-u_{2,l}+i/2}.
\end{aligned} \tag{5.20}$$

The energy eigenvalues are given by

$$E^{(2)}_{SO(6)} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{M_2}\frac{1}{u_{2,i}^2+1/4} \tag{5.21}$$

Alternatively, the Bethe equations (5.20) can be thought of as equations for vector representation $\mathfrak{so}(6)$ with simple roots $\epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \epsilon_2+\epsilon_3$ and weight $w=\epsilon_1$, or for the antisymmetric representation of $\mathfrak{su}(4)$ with roots $\epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3, \epsilon_3-\epsilon_4$ and weight $w=\epsilon_1+\epsilon_2$. The corresponding Dynkin diagram is depicted in Fig. 5.4.

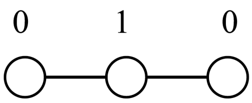


Figure 5.4 Dynkin diagram and the Dynkin labels $\alpha_r \cdot w$ corresponding to the $\mathfrak{so}(6)$ Bethe ansatz.

The $SU(2)$ sector[36] is a subsector of $SO(6)$, where there are only spin up ($\uparrow$) and spin down ($\downarrow$). The $SU(2)$ sector Hamiltonian is

$$H^{(2)}_{SU(2)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1 - P_{l,l+1}\right), \tag{5.22}$$

since there is no contribution from $K_{l,l+1}$. In terms of spin operators the Hamiltonian can be recast as

$$H^{(2)}_{SU(2)} = \frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(\frac{1}{2} - 2\vec{S}_l\cdot\vec{S}_{l+1}\right). \tag{5.23}$$

Remarkably, $\Gamma_{SU(2)}$ is the Hamiltonian of the Heisenberg $XXX_{1/2}$ spin-chain with L lattice sites (cf. (3.15)). The total spin $\vec{S} = \sum_l \vec{S}_l$ commutes with Γ so the energy eigenstates are simultaneously the total spin eigenstates. This should not be surprising since we have known that the dilatation operator commutes with the R-symmetry and the spin here is one of its subgroups.

The $SU(2)$ Bethe equations can be obtained from (5.20) by restricting $M_1 = M_3 = 0$ and $M_2 = M$. Due to the cyclicity of the trace in the local operators in the gauge theory, we have to retain only the states of the spin chain with zero total momentum, i.e. invariant by translation. This extra condition can be written as

$$\prod_{k=1}^{M_2}\frac{u_{2,k}+i/2}{u_{2,k}-i/2} = 1. \tag{5.24}$$

Extending to the full $\widetilde{PSU}(2,2|4)$ sector, its one-loop Hamiltonian was found by Beisert [12] and the full one-loop Bethe ansatz was written in [19] with the Bethe equations in a compact form,

$$\left(\frac{u_{i,k}+\frac{i}{2}V_k}{u_{i,k}-\frac{i}{2}V_k}\right)^L = \prod_{\substack{l=1\\ l\neq k}}^{M_r}\prod_{j=1}^{J_l}\frac{u_{i,k}-u_{j,l}+\frac{i}{2}\mathrm{M}_{kl}}{u_{i,k}-u_{j,l}-\frac{i}{2}\mathrm{M}_{kl}} \tag{5.25}$$

where $k = 1,\dots,r = \mathrm{rank}(\mathfrak{psu}(2,2,|4))$ and $i = 1,\dots,J_k$ with J_k being the number of

[36] $SU(2)$ sector is closed to all orders in perturbation theory under operator mixing.

excitations of type k (each type corresponds to a different node of the Dynkin diagram; hence k has rank($\mathfrak{psu}(2,2,|4)$) possible values) and there are $J = \sum J_k$ excitations in total. M_{kl} denotes the symmetric Cartan matrix of $\mathfrak{psu}(2,2|4)$,

$$\mathrm{M}_{kl} = \left(\begin{array}{c|c|ccc|c|c} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & +2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & +2 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right) \tag{5.26}$$

and the highest weights for the representation are all 0 except for $V_4 = 1$, i.e. the spin representation vector is $V = [0,0,0,1,0,0,0]$ at the respective spin chain sites. The full one-loop Bethe equations correspond to the Dynkin diagram in Fig. 5.5.

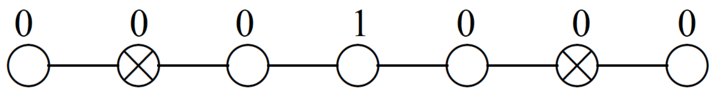


Figure 5.5 *Beauty* Dynkin diagram corresponding to the $\mathfrak{psu}(2,2|4)$ Bethe ansatz. The Dynkin labels of the spin representation are indicated on top of the beauty diagram. The nodes $\otimes$ correspond to the fermionic roots.

Explicitly, the full one-loop Bethe equations are given by

$$
\begin{aligned}
1 &= \prod_{l\neq k}^{M_1} \frac{u_{1,k}-u_{1,l}-i}{u_{1,k}-u_{1,l}+i} \prod_{l=1}^{M_2} \frac{u_{1,k}-u_{2,l}+i/2}{u_{1,k}-u_{2,l}-i/2} \\
1 &= \prod_{l=1}^{M_1} \frac{u_{2,k}-u_{1,l}+i/2}{u_{2,k}-u_{1,l}-i/2} \prod_{l=1}^{M_3} \frac{u_{2,k}-u_{3,l}-i/2}{u_{2,k}-u_{3,l}+i/2} \\
1 &= \prod_{l=1}^{M_2} \frac{u_{3,k}-u_{2,l}-i/2}{u_{3,k}-u_{2,l}+i/2} \prod_{l\neq k}^{M_3} \frac{u_{3,k}-u_{3,l}+i}{u_{3,k}-u_{3,l}-i} \prod_{l=1}^{M_4} \frac{u_{3,k}-u_{4,l}-i/2}{u_{3,k}-u_{4,l}+i/2} \\
\left(\frac{u_{4,k}+i/2}{u_{4,k}-i/2}\right)^L &= \prod_{l=1}^{M_3} \frac{u_{4,k}-u_{3,l}-i/2}{u_{4,k}-u_{3,l}+i/2} \prod_{l\neq k}^{M_4} \frac{u_{4,k}-u_{4,l}+i}{u_{4,k}-u_{4,l}-i} \prod_{l=1}^{M_5} \frac{u_{4,k}-u_{5,l}-i/2}{u_{4,k}-u_{5,l}+i/2} \\
1 &= \prod_{l=1}^{M_4} \frac{u_{5,k}-u_{4,l}-i/2}{u_{5,k}-u_{4,l}+i/2} \prod_{l\neq k}^{M_5} \frac{u_{5,k}-u_{5,l}+i}{u_{5,k}-u_{5,l}-i} \prod_{l=1}^{M_6} \frac{u_{5,k}-u_{6,l}-i/2}{u_{5,k}-u_{6,l}+i/2} \\
1 &= \prod_{l=1}^{M_5} \frac{u_{6,k}-u_{5,l}-i/2}{u_{6,k}-u_{5,l}+i/2} \prod_{l=1}^{M_7} \frac{u_{6,k}-u_{7,l}+i/2}{u_{6,k}-u_{7,l}-i/2} \\
1 &= \prod_{l=1}^{M_6} \frac{u_{7,k}-u_{6,l}+i/2}{u_{7,k}-u_{6,l}-i/2} \prod_{l\neq k}^{M_7} \frac{u_{7,k}-u_{7,l}-i}{u_{7,k}-u_{7,l}+i}
\end{aligned}
\tag{5.27}
$$

with energy

$$
E^{(2)}_{\text{full}} = \frac{\lambda}{8\pi^2}\sum_{i=1}^{M_4}\frac{1}{u_{4,i}^2+1/4}. \tag{5.28}
$$

To avoid Bethe roots at infinity, the magnon numbers must obey $M_1 < M_2 < M_3 < M_4 > M_5 > M_6 > M_7$.

Going beyond one-loop, one will find that the n-loop anomalous dimension can involve up to n neighbouring fields in a long range Hamiltonian corresponding to the n-loop dilatation operator [94]. The spin chain is therefore effectively long range at strong coupling. On the assumption of exact quantum integrability of the gauge theory, the full sector $\mathcal{N}=4$ SYM theory is conjectured to be integrable at *all loops* [16]. This conjecture is further studied by Beisert and Staudacher [20] who wrote down the all-loop $\mathfrak{psu}(2,2|4)$ *asymptotic Bethe equations* (ABE), complemented by the so-called *dressing factor* [17, 24]. The Bethe procedure is *asymptotic* to correctly capture the behaviour of the anomalous dimension only up to λ^L order for a chain of length L.

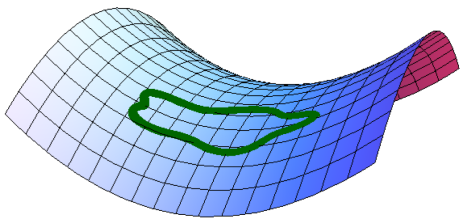
After this order, the *wrapping effects* are taken into account reflecting the fact that the chain has a finite size [62, 95]. At the order n in perturbation theory, the spin chain Hamiltonian $H_{l,l+1,\ldots,l+n}$ involves interaction up to $n+1$ sites. For a spin chain of total length $L=n+1$, it is obvious that there might be interactions that go over all the spin chain; hence they *wrap* the chain. In this case the ABE are no longer valid. In order to compute these finite-size effects, different techniques have been developed, such as the Lüscher corrections [70, 71] and the *thermodynamic Bethe Ansatz* (TBA) equations (an infinite set of coupled integral equations) [11, 28, 57]. This served as a basis for the so-called *Y-system* comprising an infinite set of nonlinear functional equations [58].

Part III

$AdS_5 \times S^5$ Superstring Theory and Its Integrability

Chapter 6

Type IIB Superstring Theory on $AdS_5 \times S^5$



A closed string on curved spacetime.

6.1 Strings and $AdS_5 \times S^5$ Spacetime

In the 1970s, string theory itself was discovered in the attempt to describe the hadronic physics [105]. However, the theory was dismissed soon after the advent of Quantum Chromodynamics (QCD) as the precise theory of the strong nuclear force. It was realised in the late 1990s that the dualities between gauge and (super)string theory managed to circumvent the difficulties physicists faced in the last two decades in applying superstring theory to strong interaction.

One of the maximally supersymmetric *vacua* (solutions) admitted by the type IIB superstring theory is the product of the five-dimensional anti-de Sitter space AdS_5 and the five-sphere S^5, as illustrated in Fig. 6.1. This $AdS_5 \times S^5$ spacetime is supported by the self-dual Ramond-Ramond (RR) five-form flux. The other two solutions are the limits of this spacetime – the flat ten-dimensional Minkowski space and the plane-wave background [26]. Below we shall have a look at the definition of $AdS_5 \times S^5$ spacetime.

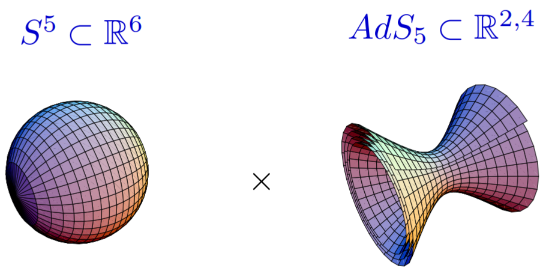


Figure 6.1 An artist's impression of the topological structure of $AdS_5 \times S^5$ spacetime. AdS_5 is equivalent to a hyperbolic space but with Minkowski signature.

Taking an analogy to the 5-sphere S^5 of radius R that can be embedded as a surface in $\mathbb{R}^6$,

$$\delta_{MN} X^M X^N = X_1^2 + X_2^2 + \ldots + X_6^2 = R^2, \quad M, N = 1, \ldots, 6 \tag{6.1}$$

with the Euclidean metric

$$(ds^2)_{S^5} = \delta_{MN} dX^M dX^N, \quad \delta_{MN} = (+1, +1, \ldots, +1, +1), \tag{6.2}$$

the 5-dimensional anti-de Sitter space AdS_5 of radius R can be embedded in $\mathbb{R}^{2,4}$ as a hyperboloid (a constant negative curvature quadric),

$$-\eta_{PQ} Y^P Y^Q = Y_0^2 - Y_1^2 - \ldots - Y_4^2 + Y_5^2 = R^2, \quad P, Q = 0, \ldots, 5 \tag{6.3}$$

with the metric

$$(ds^2)_{AdS_5} = \eta_{PQ} dY^P dY^Q, \quad \eta_{PQ} = (-1, +1, \ldots, +1, -1). \tag{6.4}$$

We set the radius of the sphere and the hyperboloid to 1 in the following discussion.

The full $AdS_5 \times S^5$ metric is $(ds^2)_{AdS_5 \times S^5} = (ds^2)_{AdS_5} + (ds^2)_{S^5}$, where

$$\begin{aligned} (ds^2)_{AdS_5} &= d\rho^2 - \cosh^2 \rho dt^2 + \underbrace{\sinh^2 \rho \left(d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2 \right)}_{d\Omega^2_{S^3}}, \\ (ds^2)_{S^5} &= d\gamma^2 + \cos^2 \gamma d\varphi_3^2 + \underbrace{\sin^2 \gamma \left(d\psi^2 + \cos^2 \psi d\varphi_1^2 + \sin^2 \psi d\varphi_2^2 \right)}_{d\Omega^2_{S^3}}, \end{aligned} \tag{6.5}$$

which can be obtained by the parametrisation of $AdS_5 \times S^5$ coordinates in terms of $5 + 5$

independent *global coordinates*, i.e. $X_M = (\rho, \theta, \phi_1, \phi_2, t)$ and $Y_P = (\gamma, \psi, \varphi_1, \varphi_2, \varphi_3)$,

$$\begin{aligned} \widetilde{X}_1 &\equiv X_1 + iX_2 = \sin\gamma\, \cos\psi\, e^{i\varphi_1}, \quad \widetilde{X}_2 \equiv X_3 + iX_4 = \sin\gamma\, \sin\psi\, e^{i\varphi_2}, \\ \widetilde{X}_3 &\equiv X_5 + iX_6 = \cos\gamma\, e^{i\varphi_3}, \quad \widetilde{Y}_0 \equiv Y_5 + iY_0 = \cosh\rho\, e^{it}, \\ \widetilde{Y}_1 &\equiv Y_1 + iY_2 = \sinh\rho\, \cos\theta\, e^{i\phi_1}, \quad \widetilde{Y}_2 \equiv Y_3 + iY_4 = \sinh\rho\, \sin\theta\, e^{i\phi_2}. \end{aligned} \tag{6.6}$$

The global coordinates are chosen to be $\rho > 0$, $0 < t \le 2\pi$ and the S^3 angles $0 < \theta < 2\pi, 0 < \phi_1 < \pi, 0 < \phi_2 < \pi$ covering the hyperboloid once. Near the centre ($\rho = 0$) the AdS_5 metric is that of $\mathbb{R}^4 \times S^1$, whereas near its boundary ($\rho \to \infty$) it is that of $S^1 \times S^3$. To relate the theory to the corresponding gauge theory in $\mathbb{R} \times S^3$, the t direction is often decompactified to $-\infty < t < \infty$. The global coordinates in the S^5 metric follow the same periodicities as those in the AdS_5 metric correspondingly.

Another useful choice of $AdS_5 \times S^5$ coordinates in the solution of open strings ending at the AdS boundary [5, 40, 88] is the *Poincaré coordinates*. These coordinates cover only part of AdS_5 [4],

$$\begin{aligned} Y_0 &= \frac{x_0}{z} = \cosh\rho\, \sin t, \qquad Y_5 = \frac{1}{2z}\left(1 + z^2 - x_0^2 + x_i^2\right) = \cosh\rho\, \cos t, \\ Y_i &= \frac{x_i}{z} = n_i \sinh\rho, \qquad Y_4 = \frac{1}{2z}\left(-1 + z^2 - x_0^2 + x_i^2\right) = n_4 \sinh\rho. \end{aligned} \tag{6.7}$$

Here $n_i^2 + n_4^2 = 1\,(i = 1,2,3)$ parametrises the 3-sphere in AdS_5 metric, $dn_k dn_k = d\Omega_3^2\,(\theta, \phi_1, \phi_2)$. Therefore, the $AdS_5 \times S^5$ metric (6.5) takes the conformally-flat form,

$$\left(ds^2\right)_{AdS_5 \times S^5} = \frac{1}{z^2}\left(dx^m dx_m + dz_M dz_M\right), \tag{6.8}$$

where $x_m = \eta_{mn} x^n\,(m, n = 0, 1, 2, 3)$, $z^2 = z_M z_M\,(M = 1, \ldots, 6)$, and $dz_M dz_M = dz^2 + z^2 d\Omega_5^2\,(\gamma, \psi, \varphi_1, \varphi_2, \varphi_3)$.

6.2 The Holographic Principle

Let us digress to the AdS_5/CFT_4 duality prior to the discussion of superstring motion in the $AdS_5 \times S^5$ background. The discovery of the D-branes (Dirichlet-membranes[37]) [31] in string theory gave a new impetus to the study of string dualities and nonperturbative effects, and aided in understanding the relation with gauge theory. Moreover, Dp-branes[38]

[37] The name refers to the fact that open strings that end on D-branes have *Dirichlet* boundary conditions, which means that the string endpoints are stuck on the branes.

[38] A D-brane of a specific dimension is called a Dp-brane, where p indicates that the brane spans p spatial dimensions, and thus has a $(p + 1)$ dimensional spacetime worldvolume.

being solitons are extended objects on which the strings can start or end.

It was understood by Witten [106] that a stack of N coincident Dp-branes is associated to a nonabelian $U(N)$ gauge theory. In particular, for D3-branes, the near horizon geometry turns out to be $AdS_5 \times S^5$ (Fig. 6.2) and the low-energy dynamics on their worldvolume is governed by a $U(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry; hence the two computations gave exactly the same answer.

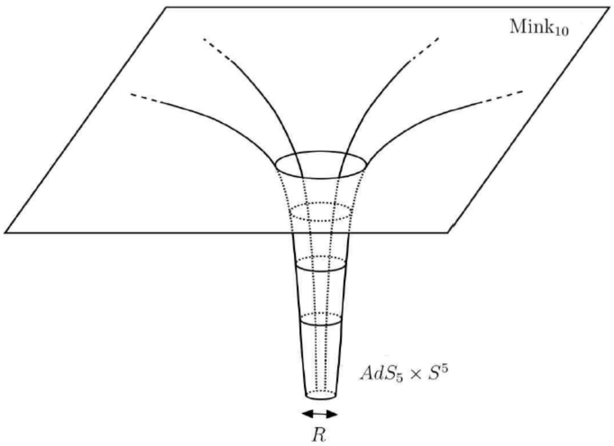


Figure 6.2 D3-branes carry mass, energy and charge, and can deform the spacetime around them to make a curved geometry. Located at the infinite end of the throat (described by $AdS_5 \times S^5$ metric) is the N D3-brane stack, known as the *horizon* (black hole location). The geometry near the D3-branes is surrounded by S^5.

Building on these ideas, Maldacena [72] conjectured in 1997 that

$\mathcal{N} = 4$ SYM in flat space $\mathbb{R}^{1,3}$
with YM coupling constant g_{YM} and gauge group $SU(N)$

is exactly dual to

type IIB superstring theory with string tension T and coupling constant g_s
on $AdS_5 \times S^5$ with curvature radii $R_{AdS_5} = R_{S^5} \equiv R$
and N units of RR five-form flux F_5 through S^5.

He considered strings propagating in the $AdS_5 \times S^5$ geometry created by a stack of N coincident D3-branes, such that the theory on the branes is the $\mathcal{N} = 4$ SYM theory. The four-dimensional spacetime can be recovered as the boundary of the AdS_5 spacetime (as in Fig. 6.3), while S^5 is associated with the internal symmetry of the gauge fields. This conjecture surprisingly relates a lower dimensional gauge theory to a higher dimensional string model, which represents a manifestation of the *holographic principle* [99, 102].

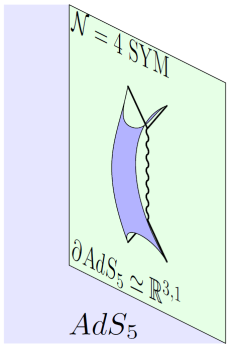


Figure 6.3 CFT_4 ($\mathcal{N}=4$ SYM theory) on boundary of AdS_5.

Just like the gauge theory is controlled by two parameters: the rank N of the gauge group and the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$, the dual string model is parametrised by the string coupling constant g_s and the effective string tension[39] $T = R^2/2\pi\alpha' = R^2/2\pi l_s^2$, where R is the common radius of the AdS_5 and S^5 geometries. These two sets of parameters are identified as

$$g_s = \frac{g_{\text{YM}}^2}{2\pi}, \quad T^2 = \frac{\lambda}{2\pi^2}. \tag{6.9}$$

Recall that in Section 4.3, the structure of the genus g expansion of Feynman diagrams in the gauge theory resembles the perturbative expansion of string theory as a sum over worldsheet amplitudes in terms of powers $g_s^{-\chi}$ where $\chi = 2-2g$ is the Euler character of the string worldsheets, as shown in Fig. 6.4. Here we have the *strongest version* of the AdS_5/CFT_4 correspondence[40] which states that the equivalence between the both theories holds for arbitrary values of N and λ.

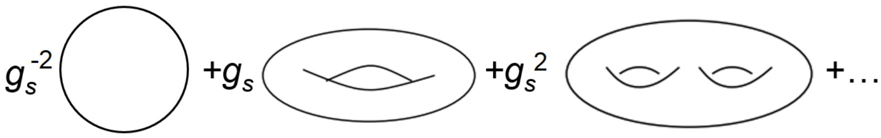


Figure 6.4 String perturbation theory.

[39] $\alpha' = l_s^2$ is the square of the fundamental string length.

[40] A better picture to visualise the AdS_5/CFT_4 duality is via the calculation of *Wilson loops*. The contour of the SYM Wilson loop represents the ends of an open string attached to the boundary of $AdS_5 \times S^5$ extending into the bulk of the AdS_5 space. The expectation value of the Wilson loop operator is semiclassically the minimal surface of the AdS_5 string, which due to the curvature extends into the bulk.

The leading order in the expansion contains string worldsheets which are topologically equivalent to the two-sphere. As discussed in Section 4.3, the nonplanar graphs are suppressed in the 't Hooft limit. Therefore, we expect that the gauge theory at $N \to \infty$ is dual to a noninteracting (or tree-level) string model, i.e. there is no string coupling ($g_s = 0$). Moreover, there is no string splitting or joining while the worldsheet coupling λ remains (Fig. 6.5). This limit gives a more *modest version* of AdS_5/CFT_4 correspondence which claims the duality in the planar limit $N \to \infty$ with λ fixed at small values.

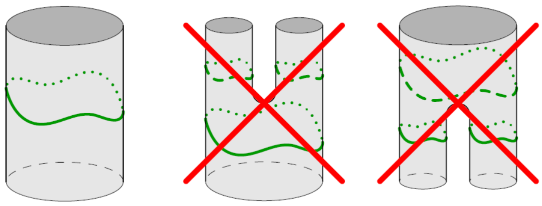


Figure 6.5 Conditions for string theory in planar limit.

Consider an arbitrary state $|\mathcal{O}\rangle$ of the string theory and the eigenvalue $E\,(g_s, T)$ of this state with respect to the string Hamiltonian H, defined as an operator conjugated to the AdS_5 time variable,

$$H|\mathcal{O}\rangle = E\,(g_s, T)\,|\mathcal{O}\rangle. \tag{6.10}$$

Considering that the string and gauge theory share the same symmetry algebra and representations, one can conclude that the string Hamiltonian H corresponds to the dilatation operator D on the gauge theory side. Therefore, there should exist an eigenstate $\mathcal{O}(x)$ such that

$$D\mathcal{O}(x) = \Delta(g_{\mathrm{YM}}, N)\mathcal{O}(x), \quad \Delta(g_{\mathrm{YM}}, N) = E\,(g_s, T)\,. \tag{6.11}$$

In the planar limit ($N \to \infty$), the correspondence reduces to

$$\Delta(g_{\mathrm{YM}}, \infty) = E\,(0, T)\,. \tag{6.12}$$

This statement simply translates as the scaling dimensions of the planar gauge theory is identical with the energies of the free string theory!

Unfortunately though, even the free ($g_s = 0$) $AdS_5 \times S^5$ string is a rather complicated 2D field theory, whose quantisation remains a very challenging open problem; thus a direct verification of (6.12) is not feasible. We can, however, address the string theory side by studying its low energy effective description in terms of type IIB supergravity (SUGRA). This approximation is only meaningful as long as the curvature of the background is small compared to the string scale, i.e. the radius R in string units needs to be very large ($R \gg l_s$) or

$$1 \ll \frac{R^4}{\alpha'^2} = \frac{R^4}{l_s^4} = \lambda. \tag{6.13}$$

On the dual $\mathcal{N} = 4$ SYM theory side, this is equivalent to the strong coupling regime of the theory, and hence we arrive at the *weak version* of the correspondence. To summarise, all the three versions of the AdS_5/CFT_4 duality are depicted in Fig. 6.6.

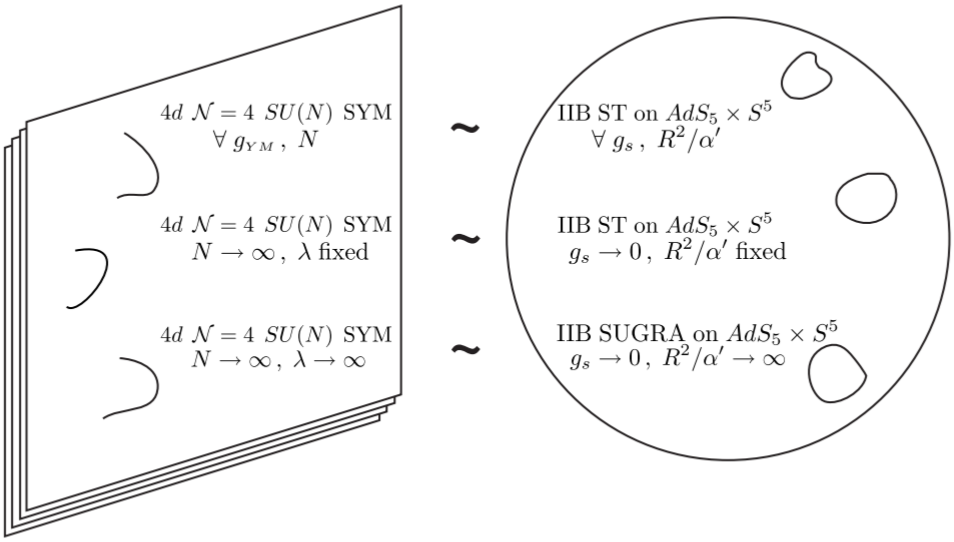


Figure 6.6 Three versions of the AdS_5/CFT_4 duality. On the left a stack of N D3-branes is hosting open strings, whereas on the right the closed string modes propagate in the AdS_5 space [7].

A *caveat* in the weak version of AdS_5/CFT_4 duality is that the perturbative domain of $AdS_5 \times S^5$ superstring, i.e. $\lambda \gg 1$, is perfectly *incompatible* to the perturbatively controllable regime of $\mathcal{N} = 4$ SYM, i.e. $\lambda \ll 1$! We are therefore dealing with a strong/weak coupling duality; proving the duality is very hard if not impossible, as it requires solving the string or gauge theory *nonperturbatively*. A summary of the duality of the observables and parameters of the both sides of the AdS_5/CFT_4 correspondence is depicted in Table 6.1.

6.3 $AdS_5 \times S^5$ Superstring Action

Since there is a nonvanishing RR 5-form flux supporting the $AdS_5 \times S^5$ background, the Neveu-Schwarz-Ramond (NSR) formalism of the superstring action [80, 84] is precluded in a straightforward way. The Green-Schwarz (GS) approach [53, 54] appears to be

Table 6.1 Dictionary of AdS_5/CFT_4 correspondences.

$\mathcal{N}=4$ **SYM**	**Strings on** $\mathbf{AdS_5 \times S^5}$
Local operators $\mathcal{O}$	String states $\lvert\mathcal{O}\rangle$
Dilatation operator D	String Hamiltonian H
Scaling dimension Δ	String mode energy E
$N \to \infty$, $\lambda \to \infty$ (strong coupling)	$g_s \to 0$, $T \to \infty$ (classical strings / SUGRA limit)
$N \to \infty$, λ fixed	$g_s \to 0$, T fixed (tree-level / free strings)
Any N, λ	Any g_s, T (quantum strings)

more viable as it endows the superstring action with invariance under supersymmetry and κ-symmetry[41]. However, it is not very practical for finding the explicit form of the action in terms of the coordinate fields[42]. A more advantageous approach, namely the supercoset formulation, will be discussed in the next chapter. For the review of some classical superstring solutions in this section, we use the former method, i.e. the coordinate field description.

As previously mentioned, the type IIB superstring theory in a curved space with an RR 5-form background is best described by the GS action [39, 45, 47, 74],

$$\begin{aligned} S = S_{\mathrm{B}} - S_{\mathrm{F}}, \quad S_{\mathrm{B}} &= \frac{T}{2}\int d\tau\, d\sigma \sqrt{-g}g^{ab}G_{\mu\nu}(x)\partial_a x^\mu \partial_b x^\nu, \\ S_{\mathrm{F}} = iT\int d\tau\, d\sigma &\left(\sqrt{-g}g^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ}\right)\bar{\vartheta}^I \rho_a D_b \vartheta^J + O\left(\vartheta^4\right) \end{aligned} \tag{6.14}$$

with the AdS_5/CFT_4 identified string tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$. Here $g_{ab}\,(a, b = 0, 1 \,\mathrm{or}\, \tau, \sigma)$ is an independent 2D metric and x^μ $(\mu = 0, 1, \ldots, 9)$ are the bosonic string coordinates. ϵ^{ab} is a 2D antisymmetric tensor with $\epsilon^{01} = 1$, ϑ^I $(I = 1, 2)$ are two Majorana-Weyl spinor fields, and $s^{IJ} = \mathrm{diag}(1, -1)$. ρ_a are projections of the 10D Dirac matrices, $\rho_a \equiv \Gamma_A E^A_\mu \partial_a x^\mu$, where E^A_μ is the vielbein of the target space metric, $G_{\mu\nu} = E^A_\mu E^B_\nu \eta_{AB}$. D_a is the projection of the 10D covariant derivative D_μ, where $D_\mu = \partial_\mu + \frac{1}{4}\omega^{AB}_\mu \Gamma_{AB} - \frac{1}{8\cdot 5!}\Gamma^{\mu_1\ldots\mu_5}\Gamma_\mu F_{\mu_1\ldots\mu_5}$ with ω^{AB}_μ as the Lorentz connection. $F_{\mu_1\ldots\mu_5}$ is the RR 5-form field, which should be related to $G_{\mu\nu}$ so that the 2D Weyl and κ-symmetry anomalies cancel.

Working with the embedding coordinates[43] $X_M(\tau, \sigma)$ and $Y_P(\tau, \sigma)$, and in the con-

[41] κ-symmetry is a local fermionic symmetry that ensures the correct number of physical fermionic degrees of freedom in a theory.

[42] In terms of the coordinate fields, one has to determine the exact expressions for the supervielbeins in a bosonic $AdS_5 \times S^5$ background. This is indeed a difficult problem to solve in nontrivial cases.

[43] As the string moves, it sweeps out a 2D surface, called a worldsheet, Σ, parametrised by the space and time coordinates, i.e. σ and τ, respectively. These coordinates are used in defining the *sigma model* – a field theory wherein fields take values in a curved manifold $\mathcal{M}$. Put differently, a field configuration

formal (or orthogonal) gauge[44], $\sqrt{-g}g^{ab} = \eta^{ab} = \mathrm{diag}(-1,1)$, the *$AdS_5 \times S^5$ bosonic* superstring action (6.14) is given by

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d\tau\, d\sigma\Big(\underbrace{-\partial_a Y_P \partial^a Y^P - \tilde{\Lambda}\left(Y_P Y^P + 1\right)}_{L_{AdS}} + \underbrace{-\partial_a X_M \partial^a X^M + \Lambda\left(X_M X^M - 1\right)}_{L_S}\Big) \tag{6.15}$$

Here, the *fermionic* terms are suppressed in the action, as they will not be relevant in the discussion of classical solutions (the full fermionic action is stated in [74, 89]). Λ and $\tilde{\Lambda}$ are the Lagrange multipliers (functions of τ and σ) imposing the two hypersurface conditions. The action (6.15) is supplemented with the conformal gauge constraints (the Virasoro conditions) expressing the vanishing of the total 2D energy-momentum tensor T_{ab} ,

$$\begin{aligned} T_{00} = T_{11} = 0 \implies \dot{Y}_P \dot{Y}^P + Y'_P Y'^P + \dot{X}_M \dot{X}^M + X'_M X'^M = 0, \\ T_{01} = T_{10} = 0 \implies \dot{Y}_P Y'^P + \dot{X}_M X'^M = 0, \end{aligned} \tag{6.16}$$

where $\cdot$ and $'$ denote temporal derivative and spatial derivative, respectively. The classical equations of motion for (6.15) are

$$\begin{aligned} \partial^a\partial_a Y_P - \tilde{\Lambda} Y_P = 0, \quad \tilde{\Lambda} = \partial^a Y_P \partial_a Y^P, \qquad Y_P Y^P = -1, \\ \partial^a\partial_a X_M + \Lambda X_M = 0, \quad \Lambda = \partial^a X_M \partial_a X^M, \qquad X_M X^M = 1. \end{aligned} \tag{6.17}$$

We will be interested in the closed string solutions with the worldsheet as a cylinder (Fig. 6.7) for which the periodicity conditions are imposed,

$$Y_P(\tau, \sigma + 2\pi) = Y_P(\tau,\sigma), \quad X_M(\tau,\sigma+2\pi) = X_M(\tau,\sigma). \tag{6.18}$$

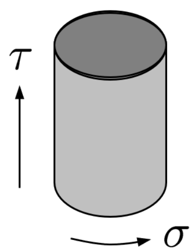


Figure 6.7 Closed string worldsheet as a cylinder.

The action (6.15) is invariant under the $SO(2,4)$ and $SO(6)$ rotations with the

is a map $X^\mu(\sigma) : \Sigma \to \mathcal{M}$, where $\mathcal{M}$ is known as the target space.

[44] η^{ab} is the Weyl-invariant combination of the worldsheet metric g^{ab} with $\det \eta = -1$.

conserved (on-shell) bosonic charges

$$S_{PQ} = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \left(Y_P \dot{Y}_Q - Y_Q \dot{Y}_P \right), \quad J_{MN} = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \left(X_M \dot{X}_N - X_N \dot{X}_M \right). \quad (6.19)$$

The isometry group of the $AdS_5 \times S^5$ metric (6.5) is $SO(2,4) \times SO(6)$, which has $3+3$ linear isometries, i.e. the translations in the time t, in the 2 angles ϕ_a and the 3 angles φ_i. A natural choice of the corresponding $3+3$ Cartan generators[45] of the isometry group is

$$\begin{aligned} S_0 \equiv S_{50} \equiv E = \sqrt{\lambda}\mathrm{E}, \qquad S_1 \equiv S_{12} = \sqrt{\lambda}\mathrm{S}_1, \qquad S_2 \equiv S_{34} = \sqrt{\lambda}\mathrm{S}_2, \\ J_1 \equiv J_{12} = \sqrt{\lambda}\mathrm{J}_1, \qquad J_2 \equiv J_{34} = \sqrt{\lambda}\mathrm{J}_2, \qquad J_3 \equiv J_{56} = \sqrt{\lambda}\mathrm{J}_3. \end{aligned} \quad (6.20)$$

We shall restrict to classical solutions (or states) characterised by finite values of the AdS (worldsheet) energy E, the spins $S_r\,(r = 1,2)$ on AdS_5, and the angular momenta $J_i\,(i = 1,2,3)$ on S^5. Given the Virasoro conditions, the energy can be expressed in terms of the other five charges,

$$E = \sqrt{\lambda}\mathrm{E}\left(\mathrm{S}_r, \mathrm{J}_i; k_s\right) = \sqrt{\lambda}\mathrm{E}\left(\frac{S_r}{\sqrt{\lambda}}, \frac{J_i}{\sqrt{\lambda}}; k_s\right) \quad (6.21)$$

with other (hidden) conserved charges k_s, like “topological” numbers determining particular shape of the string (e.g., number of spikes, folds, winding numbers, etc).

6.4 Classical Solutions

To find exact solutions of the nonlinear equations of motions, one typically makes an ansatz for the shape of the string. Here we consider the simplest (ground-state) case – point-like string, for which it is localised at the centre of AdS_5, i.e. $Y_P = Y_P(\tau), X_M = X_M(\tau)$ in (6.16) and (6.17). This particular string has a *geodesic* in $AdS_5 \times S^5$. The natural implication from (6.16) and (6.17) is that $\Lambda, \tilde{\Lambda} =$ constant with $\Lambda = -\tilde{\Lambda} > 0$.

The solution for a massless geodesic that lies entirely within AdS_5 is a straight line in $\mathbb{R}^{2,4}$,

$$Y_P(\tau) = A_P + B_P \tau, \quad B_P B^P = A_P B^P = 0, \quad A_P A^P = -1. \quad (6.22)$$

The $SO(2,4)$ spin tensor in (6.19) is $S_{PQ} = \sqrt{\lambda}\,(A_P B_Q - A_Q B_P)$. This solution does not

[45] The 6 charges are exactly the unitary representations of the bosonic subgroup in the gauge theory side (cf. Section 4.4) with the AdS_5/CFT_4 identification of the AdS energy $E = \Delta$.

represent a "highest-weight" semiclassical state[46] since there are always nonvanishing non-Cartan components, e.g., if $Y_5+iY_0 = 1+ip\tau$, $Y_3 = p\tau$, $Y_{1,2,4} = 0$ we get $S_{50} = S_{53} = \sqrt{\lambda}p$.

There is another point-like string in $AdS_5 \times S^5$ that runs along the time direction in AdS_5 and carries one component of large momentum $J = J_3 \gg 1$ wrapping the equator (or great circle) of S^5. The geometry seen by this fast moving string is the plane wave geometry, and string theory in this background is exactly solvable. The angular motion in S^5 provides an effective mass to a particle in AdS_5, i.e. the corresponding geodesic in AdS_5 is a massive one,

$$Y_5 + iY_0 = e^{i\kappa\tau}, \quad X_5 + iX_6 = e^{i\kappa\tau}, \quad \kappa = \sqrt{\Lambda}, \quad Y_{1,2,3,4} = X_{1,2,3,4} = 0. \tag{6.23}$$

The only nonvanishing integrals of motion are the energy and the $SO(6)$ angular momentum of this BPS state[47], $E = J_3 = \sqrt{\lambda}\kappa$, matching the minimal scaling dimension Δ of the chiral primary operator (also known as Berenstein-Maldacena-Nastase (BMN) "vacuum" operator) tr $\left(Z^{J_3}\right)$, $Z = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$, in the gauge theory [22].

In general, a string of which all points moving fast in S^5 will admit a BMN-type "fast string" limit, i.e. AdS energy E depends analytically on the square of string tension T or on λ when expressed in terms of S_r and J_i and expanded in large total angular momentum of S^5 [47, 48]. For a string whose centre is at rest or which moves only within the AdS_5, its energy will have nonanalytic dependence on $\sqrt{\lambda}$ [48, 60, 82].

Progresses have been made on non-BPS states by concentrating on a particular class of extended (σ-dependent) string solutions of the equations (6.17) subject to the constraints (6.16), (6.18) that have finite AdS energy. In particular, *rigid* closed string solutions (i.e. the shape of the string does not change with time) have been mostly discussed. For the case of two nonvanishing angular momenta (J_1, J_2) the string evolution equations are solved in terms of elliptic functions; the corresponding string configurations can have *folded* [50] or *circular* [10, 48] profiles[48] (Fig. 6.8), giving rise to two different expressions for the energy. Both profiles have shown to agree with their SYM counterparts with the

[46] A solution having a consistent *semiclassical* interpretation should correspond to a state of a quantum Hamiltonian H which carries the same quantum numbers, and should thus be associated to a particular SYM operator with definite scaling dimension. It should be a representation of a "highest-weight" state of a symmetry algebra, i.e. all other non-Cartan (noncommuting) components of the symmetry generators (6.19) should vanish; other members of the multiplet can be obtained by applying rotations to a "highest-weight" solution.

[47] Recall that BPS state is the "protected" chrial primary state which does not change with the coupling constant (cf. Section 4.4). The point-like string having only one nonzero component of J is the only BPS state.

[48] "Folded" means that the string has a shape of an interval, while "circular" string has a shape of a circle, as the name suggested.

relevant Bethe solutions and associated scaling dimensions found in [15, 18].

A perfect agreement between string energies and scaling dimensions of gauge theory operators is also shown in a pulsating [49] string [75] and a simple circular string solution with three angular momenta (J_1, J_2, J_3) [42]. Other solutions include a folded closed string rotating in a plane in AdS_5 carries single spin, $S = S_1$ [32, 60], and also carrying two charges (S, J) [47] by boosting the centre of mass of the string rotating in AdS_5 along a circle of S^5. An interpolating solution with the three charges (S, J_1, J_2) was constructed in [90]. Further, it was realised that while extended string solutions with more spins in either or both AdS_5 and S^5 spaces may exist, they may be difficult to construct explicitly, and their AdS/CFT interpretation may be unclear.

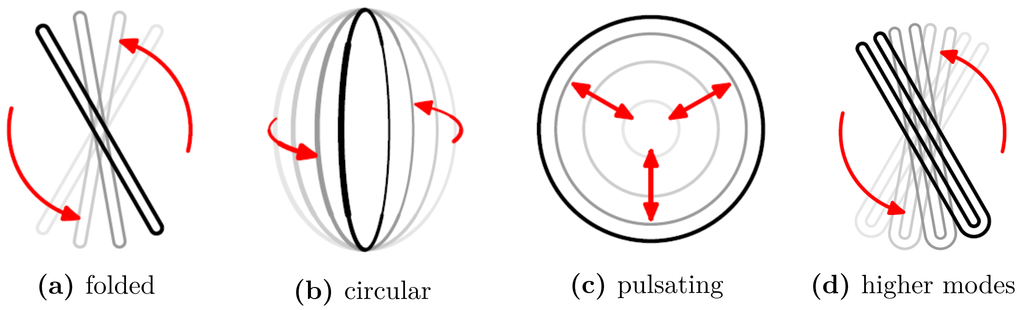


Figure 6.8 Different string configurations for rigid extended string solutions.

While the full energy spectrum of the quantum string in $AdS_5 \times S^5$ is hard to determine, some of its sections can be probed by making the ansatz of semiclassical string configurations. For multispin string states (with at least one large S^5 angular momentum component J_i), the string energy is given by its classical expression as a regular expansion in the *BMN coupling constant* $\lambda' = \lambda/J^2$, for which quantum superstring sigma model corrections are suppressed in the *BMN limit* $J \to \infty, \lambda' = \text{fixed}$ [49]. A precise test of the AdS_5/CFT_4 duality can therefore be carried out in a non-BPS sector by comparing the $\lambda' \ll 1$ expansion of the classical string energy with the corresponding quantum anomalous dimensions in perturbative $\mathcal{N} = 4$ SYM theory.

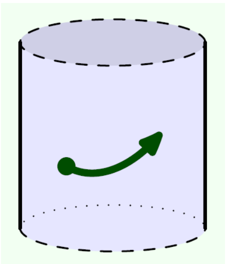
One would expect a close connection between special classes of string solutions representing particular semiclassical string states and certain integrable models. Indeed, the folded rotating string solutions with one [32, 60] or two [47, 50] nonvanishing angular momenta are related to the 1D *sine-Gordon model*. When all the three Cartan components of the SO(6) angular momentum are nonzero, the SO(6) sigma model effectively reduces

[49]The solution for pulsating string is formally not rigid but is very similar, i.e. its shape remains circular while its radius changes with time. An example of a nonrigid solution would be a "kinky" string [73] which has a shape of a quadrangle at the initial moment in time, then shrinks to diagonal due to the tension, then expands back, etc.

to a special integrable 1D *Neumann model* [79], describing a 3D harmonic oscillator with three different frequencies constrained to move on S^2. The reduction of the $AdS_5 \times S^5$ string sigma model to an integrable 1D system simply illustrates the *integrability* of this 2D theory.

Chapter 7

Classical Integrability of $AdS_5 \times S^5$ Superstring Theory



A flat connection on a coset superspace sigma model.

7.1 Supercoset Sigma Model

Although the Green-Schwarz formalism realise the spacetime supersymmetry in a manifest way such that the superstring action is invariant with respect to the κ-symmetry, the construction of the Green-Schwarz action for an arbitrary supergravity solution is difficult. Namely, one has to determine the full structure of the type IIB superfield from a given bosonic solution; a problem that has not been solved so far for a generic background.

Fortunately, the Green-Schwarz superstring action (6.14) in the $AdS_5 \times S^5$ background can be defined via a covariant sigma model on a coset superspace due to Metsaev and Tseytlin [74]. Following the construction of superstring action in the ten dimensional flat space by Henneaux and Mezincescu [61], this supercoset formulation makes use of the symmetry properties of the background solution. The group of superisometries (Killing vectors and Killing spinors) of this background is $\widetilde{PSU}(2,2|4)$, which matches the $\mathcal{N}=4$

superconformal group. The superstring action can be constructed, as a nonlinear sigma model[50] with target space being the supercoset

$$G/H = \frac{\widetilde{PSU}(2,2|4)}{SO(4,1)\times SO(5)} \simeq AdS_5\times S^5\times \text{fermions}. \tag{7.1}$$

Notice that this supercoset is not a symmetric space, as the denominator group is too small. In contrast, the bosonic part of the supercoset is a symmetry space $SO(2,4)/SO(4,1)\times SO(6)/SO(5)\simeq AdS_5\times S^5$. This supercoset model provides a natural way to couple the string worldsheet to the RR (fermionic) fields via the Wess-Zumino (WZ) term.

The Lie superalgebra $\mathfrak{g}=\mathfrak{psu}(2,2|4)$ is a quotient algebra since it is obtained by quotienting $\mathfrak{su}(2,2|4)$ over the $\mathfrak{u}(1)$ factor corresponding to the identity[51]. It admits an order-four automorphism[52], which is a linear map $\Omega:\mathfrak{g}\to\mathfrak{g}$ with

$$\Omega([a,b]) = [\Omega(a),\Omega(b)],\quad a,b\in\mathfrak{g},\quad \Omega^4=\mathbb{1} \tag{7.2}$$

that decomposes $\mathfrak{g}$ into a direct sum of four graded subspaces

$$\mathfrak{g}=\mathfrak{g}^{(0)}\oplus\mathfrak{g}^{(1)}\oplus\mathfrak{g}^{(2)}\oplus\mathfrak{g}^{(3)}, \tag{7.3}$$

where each of them is an eigenspace of Ω,

$$\Omega\left(\mathfrak{g}\right)=i^k\mathfrak{g}^{(k)},\quad k=0,1,2,3. \tag{7.4}$$

[50]The sigma model being considered here is a principal chiral model, which takes values in a group manifold G, $f(x):\Sigma\to G$. Classically This model is conformally invariant at the classical level but it develops a mass gap after quantisation.

[51]Notice that $\mathfrak{g}$ can be realised as a $4|4\times 4|4$ matrix superalgebra, but not as an 8×8 matrix superalgebra.

[52]The Lie superalgebra $\mathfrak{g}$ can be characterised by automorphism generated by the supertransposition,

$$M\to\Omega(M) = -KM^{st}K^{-1},\quad M\in\mathfrak{g},$$

where the supertranspose M^{st} of M is defined by

$$M^{st}=\begin{pmatrix} a^t & -\eta^t\\ \theta^t & b^t\end{pmatrix},\quad \left(M^{st}\right)^{st}=\begin{pmatrix} a & -\theta\\ -\eta & b\end{pmatrix}$$

and the matrix

$$K=\mathrm{diag}(\varsigma,\varsigma,\varsigma,\varsigma),\quad \varsigma=i^{-1}\sigma_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Therefore, the automorphism endows $\mathfrak{g}$ with a $\mathbb{Z}_4$-grading [23], i.e.

$$\mathrm{Str}\,(M_m M_n) = 0 \text{ for } M_m \in \mathfrak{g}^{(m)}, \quad M_n \in \mathfrak{g}^{(n)}, \quad m+n \neq 0 \bmod 4, \tag{7.5}$$

which is compatible with the supercommutator $\left[\mathfrak{g}^{(m)}, \mathfrak{g}^{(n)}\right] \subset \mathfrak{g}^{(m+n \bmod 4)}$. Note that $\mathfrak{g}^{(0)}$ is a subalgebra and the set of stationary points of Ω, i.e. $\mathfrak{g}^{(0)} = \mathfrak{h} = \mathfrak{so}(4,1) \oplus \mathfrak{so}(5)$. There is also an implicit $\mathbb{Z}_2$-grading for which $\mathfrak{g}^{(0)}, \mathfrak{g}^{(2)}$ are even and $\mathfrak{g}^{(1)}, \mathfrak{g}^{(3)}$ are odd subspaces.

The superstring action can be constructed in terms of the *coset representative*,

$$\begin{array}{rccl} f: & \Sigma & \to & \widetilde{PSU}(2,2|4) \\ & (\tau,\sigma) & \mapsto & f(\tau,\sigma), \end{array} \tag{7.6}$$

where (τ,σ) are coordinates on the worldsheet and $f(\tau,\sigma)$ is a periodic function, $f(\tau,\sigma+2\pi) = f(\tau,\sigma)$. We can use $f(\tau,\sigma)$ to build the one-form supercurrent A with values in the $\mathfrak{psu}(2,2|4)$ algebra,

$$A \equiv -f^{-1}df = -\left(f^{-1}\partial_\tau f\right) d\tau - \left(f^{-1}\partial_\sigma f\right) d\sigma, \tag{7.7}$$

so that it is conserved, $d * A = 0$. Based on the $\mathbb{Z}_4$-grading, the supercurrent can be decomposed accordingly as

$$A = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}. \tag{7.8}$$

It follows that the supercurrent is a flat connection, namely it has vanishing two-form curvature, $F \equiv dA - A \wedge A = 0$. The supercurrent is also invariant under the left group action $f \to cf$ of a constant $c \in G$. Under the local right action $f \to fh$ with $h(\tau,\sigma) \in H$, it transforms as $A \to h^{-1}Ah - h^{-1}dh$, which in components splits into

$$A^{(0)} \to h^{-1}A^{(0)}h - h^{-1}dh, \quad A^{(k)} \to h^{-1}A^{(k)}h, \quad k = 1,2,3. \tag{7.9}$$

The superstring action in $AdS_5 \times S^5$ (6.14) is then written as

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau\, d\sigma\, \mathrm{Str}\left(A^{(2)} \wedge *A^{(2)} - A^{(1)} \wedge A^{(3)} + \Lambda \wedge A^{(2)}\right), \tag{7.10}$$

or in terms of the supercurrent components, $A_\alpha = -f^{-1}\partial_\alpha f$,

$$S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau\, d\sigma\, \mathrm{Str}\left[\eta^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta + \kappa\epsilon^{\alpha\beta}A^{(1)}_\alpha A^{(3)}_\beta\right], \quad \alpha,\beta = 0,1. \tag{7.11}$$

The Lagrange multiplier Λ in the last term of (7.10) ensures the *supertraceless* condition of $A^{(2)}$ as is required for $\mathfrak{psu}(2,2|4)$, and also $A^{(0)}$ has been gauged away in this formulation of action. In (7.11), $\epsilon^{\alpha\beta}$ and $\eta^{\alpha\beta}$ are the tensors defined in (6.14). The first term is the usual kinetic term of a sigma model with interactions; hence it is regarded as a nonlinear sigma model on $AdS_5 \times S^5$. The second term is a WZ term which relies on the $\mathbb{Z}_4$ decomposition of $\mathfrak{g}$. The WZ term is a 2-form [23] which arises from a closed and exact 3-form,

$$2\,\mathrm{Str}\left(A^{(2)} \wedge A^{(3)} \wedge A^{(3)} - A^{(2)} \wedge A^{(1)} \wedge A^{(1)}\right) = d\,\underbrace{\mathrm{Str}\left(A^{(1)} \wedge A^{(3)}\right)}_{\text{WZ term}}. \tag{7.12}$$

The coefficient κ is in fact fixed to ± 1 in order to have κ-symmetry [9, 56].

One can find the equations of motion of $AdS_5 \times S^5$ superstring by varying the action (7.11) with respect to f,

$$\partial_\alpha S^\alpha - [A_\alpha, S^\alpha] = 0 \tag{7.13}$$

where $S^\alpha = \eta^{\alpha\beta} A^{(2)}_\beta - \frac{1}{2}\epsilon^{\alpha\beta}\left(A^{(1)}_\beta - A^{(3)}_\beta\right)$ and the supercurrent A_α is also a solution of the Maurer-Cartan equation

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \tag{7.14}$$

The equation of motion (7.13) has the identical form as the equation of conservation of the Nöther current $(\sqrt{\lambda}/2\pi) f S^\alpha f^{-1}$ associated with the $\mathfrak{psu}(2,2|4)$ symmetry. The corresponding Nöther charge and its projection onto an element $M \in \mathfrak{psu}(2,2|4)$ are respectively

$$Q = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma f S^0 f^{-1} \quad \text{and} \quad Q_M = \mathrm{Str}(QM). \tag{7.15}$$

7.2 Classical Lagrangian Integrability

Classical integrability is a bonus symmetry[53] in all supercoset sigma models with $\mathbb{Z}_4$-grading. Much like the Heisenberg spin chain, the sigma model is classically integrable if its Euler-Lagrange equations of motion can be cast into zero-curvature form, with a Lax connection $L_\alpha(\tau, \sigma, u)$ depending on the dynamical fields and on a spectral parameter $u \in \mathbb{C}$, i.e.

$$dL + L \wedge L = 0 \quad \text{or} \quad \partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0. \tag{7.16}$$

[53] This does not mean that integrability is not spoiled by quantum effects [1, 2].

This is a strong condition on the classical dynamics of the model because it has to be satisfied for any value of u. There is, however, a certain level of arbitrariness in constructing L, as reflected by the non-uniqueness of L in the gauge transformation

$$L_\alpha \to U L_\alpha U^{-1} + \partial_\alpha U U^{-1}, \quad U(\tau, \sigma + 2\pi) = U(\tau, \sigma), \tag{7.17}$$

leaving the vanishing of the "field strength" built out of L unaffected.

From the Lax connection, we form the monodromy matrix $T(u)$ by parallel-transporting L along a closed path γ encircling the spacelike σ direction of the worldsheet cylinder,

$$T(u) \equiv \mathcal{P} \exp \oint_\gamma d\sigma L_\sigma(\tau, \sigma, u) = \mathcal{P} \exp \int_0^{2\pi} d\sigma L_\sigma(\tau, \sigma, u), \tag{7.18}$$

as shown in Fig. 7.1. The flatness condition (7.16) guarantees that (7.18) is independent of the timeslice at fixed τ of such loop. The monodromy matrix T should satisfy the equation

$$\partial_\tau T(\tau) = [L_\tau(0, \tau, u), T(\tau)]\,. \tag{7.19}$$

Practically, the Taylor expansion of its eigenvalues in the continuous complex spectral parameter u generates an infinite tower of local conserved charges.

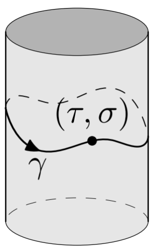


Figure 7.1 Construction of monodromy matrix in supercoset sigma model.

As pedagogically shown in [9], it requires some effort to be set up a Lax connection for the supercoset sigma models with $\mathbb{Z}_4$-grading. One can start with an ansatz for $L_\alpha(u)$ in terms of the supercurrent components $A^{(k)}$ in the constraint of the flatness condition (7.16). The superstring equations of motion (7.13) eventually lead to the zero-curvature condition for

$$L_\alpha = A^{(0)}_\alpha + \frac{1}{2}\left(u^2 + \frac{1}{u^2}\right) A^{(2)}_\alpha - \frac{1}{2\kappa}\left(u^2 - \frac{1}{u^2}\right) \eta_{\alpha\beta}\epsilon^{\beta\gamma} A^{(2)}_\gamma + u A^{(1)}_\alpha + \frac{1}{u} A^{(3)}_\alpha. \tag{7.20}$$

In literature, the corresponding (full and bosonic) Lax connection and the associated conservation laws have been studied in [6, 64]. The relation between κ-symmetry and integrability was emphasised in [21, 83].

7.3 Quantum Integrability and Comparison

In order to verify the integrability of the $AdS_5 \times S^5$ superstring theory at the quantum level, i.e. to compute the massive spectrum of the quantised (or semiclassical) worldsheet theory, one has to consider the Hamiltonian point of view. Moreover, it is worthwhile recalling how the methods of factorized scattering theory [109] are applied in the quantum case. In the following, we would like to briefly describe the multistep procedure in solving the quantum superstring sigma model.

The usual primary step for quantisation within the Green-Schwarz formulation is to go to a light-cone gauge. In such a gauge, the theory would have a physical spectrum given by the quantised light-cone Hamiltonian for a fixed light-cone momentum. However, fixing the light-cone gauge is subtle due to κ-symmetry, and hence the Hamiltonian can hardly be quantised straightforwardly.

The idea is then to study first the *decompactification* limit by considering the theory on a plane instead of a cylinder. Since the theory has a massive spectrum, the notion of asymptotic states (particles) is well defined; it thus makes sense to talk about scattering theory. Note however that the light-cone gauge action is not Lorentz invariant, and thus some properties must be adapted and extended to the case at hand. Quantum integrability should then imply the absence of particle production and factorisation of $n \to n$ worldsheet S-matrix into a product of $2 \to 2$ S-matrices (cf. Section 3.3). It is in fact not necessary to have an infinite tower of conserved quantities (see [37] for a review).

Assuming quantum integrability, the next step is to find the dispersion relation for elementary excitations and the two-body S-matrix from the (off-shell) symmetries of the light-cone gauged action in the decompactification limit. Therefore, an important question related to that program is to determine these symmetries. Once all these steps are completed, finite size effects can be considered and the findings of the light-cone superstring sigma model on a plane can be *upgraded* to a cylinder.

All the considerable effort spent to solve both theories in the AdS_5/CFT_4 duality leads to one question: *how to compare the string spectrum and the gauge spectrum?* A conventional way is to take the *sledgehammer* approach, which directly compares the classical spinning string energies to the gauge theory scaling dimensions. It is, however, a challenging task to compute the full spectrum on both theories. An alternative route for the comparison lies in the *coherent-state* effective action approach pioneered by Kruczenski [68]. The relation between phase-space action for the lightlike string and the coherent-state action on the SYM (spin chain) side gives an explicit picture of how string action "emerges" on the gauge theory side [104]. In this approach there is no need to compare explicit solutions any longer, however, considering higher-loop effects and fermions becomes more challenging in this language.

In the context of the spectral AdS/CFT correspondence, it is much more important to find a way to directly characterise the spectrum, rather than computing the explicit solutions. The idea here borrows from algebraic geometry for which an integrable system can be characterised by an *algebraic curve*. It is constructed out of the transfer matrix and encodes the information about the local charges (conserved quantities). For the classical and semiclassical string theory in $AdS_5 \times S^5$, this role is played by the *spectral curve* [92]. This would need to be compared to the curve extracted from the transfer matrix in the quantum spin chains at the thermodynamic limit.

Chapter 8

Conclusions and Outlook

This dissertation is devoted to the introductory study of the integrability of gauge and string theories. The notion of integrability was first discussed at the classical and quantum levels. Not all integrable models are logically deducible from the first principles and in certain cases it is based on clever guesses known as Bethe Ansätze. The exposition continues with the AdS_5/CFT_4 correspondence for which the integrable models behind this remarkable conjecture combines features of supersymmetry, conformal symmetry, spin chains with those of integrable sigma models.

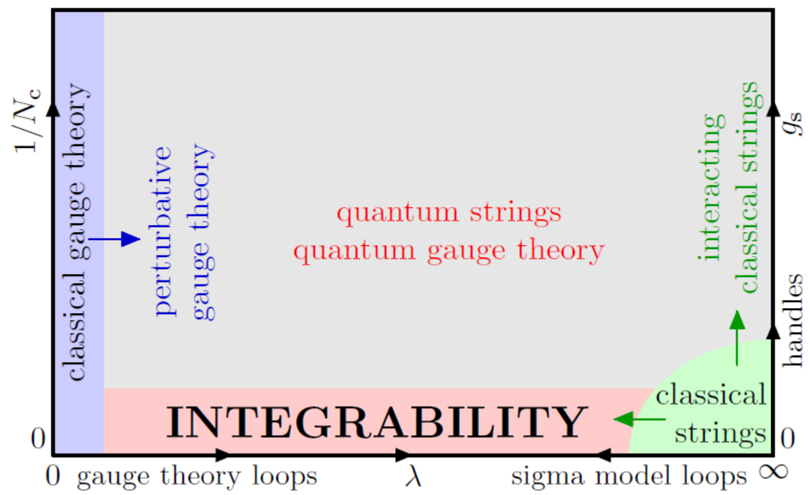


Figure 8.1 Map of the parameter space of $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ superstrings [14].

The progress due to integrability is summarised in the map of the parameter space of our gauge and string theories (Fig. 8.1). The weak coupling regime (around the point $\lambda = 0$) is where reliable results are obtained for the perturbative $\mathcal{N} = 4$ SYM theory. On the other end of the λ axis stays the strong coupling regime (around the point $\lambda = \infty$) where perturbative IIB superstring theory on $AdS_5 \times S^5$ applies. Integrability helps solve the weak/strong dilemma of the AdS_5/CFT_4 duality by restricting to planar limit (the bottom region in Fig. 8.1). It provides novel computational means in planar $\mathcal{N} = 4$ SYM at arbitrary coupling λ. The AdS_5/CFT_4 duality then relates this regime to free ($g_s = 0$) $AdS_5 \times S^5$ superstrings at arbitrary tension T.

On gauge theory side, the one-loop dilatation operator is shown to be mapped to certain sector of the integrable closed spin chain. When it was understood that integrability extends to two and three loop order in $\mathcal{N} = 4$ SYM theory, it was conjectured that integrability is an all-loop feature. On string theory side, the Metsaev-Tseytlin sigma model is shown to be classically integrable from the Lagrangian point of view. Different approaches, namely the quantum Hamiltonian and light-cone gauge, are then required to verify the integrability of semiclassical strings. Under the assumption that the string integrability persists at the quantum level, we are having a breathtaking possibility to solve the string sigma model exactly, and via the AdS_5/CFT_4 duality, to obtain an exact solution of the interacting conformal field theory in four dimensions.

In the last two decades, an arsenal of investigations have dealt with extending the applications of integrability to other observables beyond the planar spectrum and scattering amplitudes [5, 40, 67, 88, 110]. It turns out that in addition to the superconformal symmetry the special scalar integrals which enters the expression of the amplitudes obey a dual superconformal symmetry. It was argued [41] that the two sets of conformal symmetries close onto an infinite-dimensional algebra which is at the heart of integrability – the Yangian. This implied the existence of an integrable structure at the level of the amplitudes, and not only of the conformal dimensions.

Among other recent developments are the study of integrability in the deformations of the $\mathcal{N} = 4$ SYM theory and in more general models – AdS_4/CFT_3 and AdS_3/CFT_2 duality models to name a few [3, 30, 59, 65, 66, 78, 81, 98]. In line with the significant progress, one of the important questions which awaits to be solved is that of the origin of integrability and its formal proof. As one can tell, there remains a lot of open problems to be explored in the integrability programme and it has become one of the active research areas in modern physics.

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