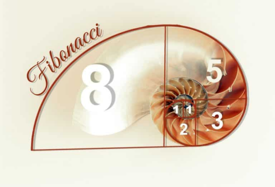
Fibonacci Anyons
&
Universal Topological Quantum Computation

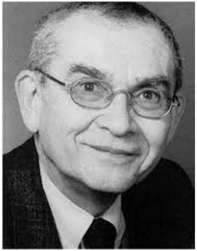
Kevin Loo

YMSC, Tsinghua University

MATHEMATICAL PHYSICS
SPRING 2021

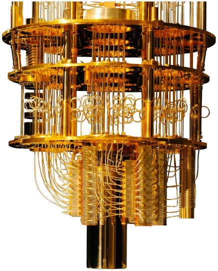


Quantum Computer



Yuri Manin

Richard Feynman



Nature isn't classical . . . and if you want to make a simulation of Nature, you'd better make it quantum mechanical . . .

—Richard Feynman (1981), *Simulating Physics with Computers*.

Topological Quantum Computer & "Any"ons

- A. Kitaev (1997): System of non-Abelian anyons with suitable properties can efficiently simulate a quantum circuit
- M. Freedman, A. Kitaev, Z. Wang (2002): System of anyons can be simulated by a quantum circuit

Is there an anyonic computational model (a topological quantum computer) that can simulate a quantum circuit that exhibits universality?

Alexei Kitaev

Michael Freedman

Fibonacci Anyons

- 2 fields: $\mathbf{1}$, τ (non-Abelian quasiparticle)
  (no field represents the underlying electron)
- A single nontrivial fusion rule:

$$\tau \times \tau = \mathbf{1} + \tau$$

- Bratteli diagram:

$$\begin{tikzcd}
 & & \tau \arrow[r] \arrow[dr] & \mathbf{1} \arrow[r] & \tau \arrow[r] \arrow[dr] & \mathbf{1} & \cdots \\
\mathbf{1} \arrow[r] & \tau \arrow[r] \arrow[ur] & \mathbf{1} \arrow[r] & \tau \arrow[r] \arrow[ur] & \mathbf{1} \arrow[r] & \tau &
\end{tikzcd}$$

- Dimension of the Hilbert space with n quasiparticles, dim$(\mathcal{H}_n)$
  $=$ number of paths through Bratteli diagram terminating at $\mathbf{1}$
  $=$ Fibonacci number $\mathrm{Fib}(n-1)$ for $n > 2$

  $$\mathrm{Fib}(1) = \mathrm{Fib}(2) = 1,$$
  $$\mathrm{Fib}(n) = \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2)$$

  Similarly, number of paths terminating at τ is $\mathrm{Fib}(n)$ for $n > 1$
  $\longrightarrow$ **Fibonacci anyon model**
- Quantum dimension of the τ particle $=$ golden mean, $d_\tau = \phi \equiv (1+\sqrt{5})/2$
  $\longrightarrow$ **Golden theory**

Fibonacci model:
the simplest known non-Abelian model that is capable of **universal topological quantum computation**
(*there exists a braid that corresponds to a unitary operation arbitrarily close to any desired operation*)

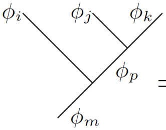
Properties of Fibonacci model:

- Hilbert space can be understood via **fusion rules** and a basis changing F-matrix
- **Braiding** of two particles can be understood as a rotation R operator that produces a phase dependent on the quantum number of the two particles

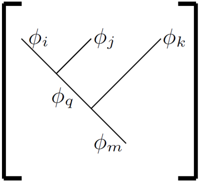
$\longrightarrow$ Encode qubits in the quantum number of some group of particles

Fusing Fibonacci Anyons

Fusion tree:

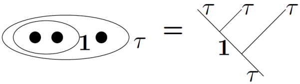


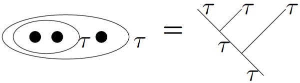
$$= \sum_q [F^{ijk}_m]_{pq}$$

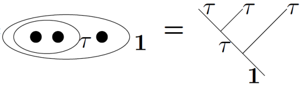


A fusion tree diagram depicts the basis states obtained by fusing fields together on different orders of fusion (although the space spanned by these states is independent of the order). The (fusion) F-matrix converts between the possible bases.

2 τ particles: $|(\bullet,\bullet)_{\mathbf{1}}\rangle$ and $|(\bullet,\bullet)_{\tau}\rangle$, with each $\bullet$ representing a τ particle
3 τ particles:

$|0\rangle = |((\bullet,\bullet)_{\mathbf{1}},\bullet)_{\tau}\rangle =$ 

$|1\rangle = |((\bullet,\bullet)_{\tau},\bullet)_{\tau}\rangle =$ 

$|N\rangle = |((\bullet,\bullet)_{\tau},\bullet)_{\mathbf{1}}\rangle =$ 

- Here, the "quantum number" of an individual particle is τ.
- The three Fibonacci particles represent a **qubit**; the three possible states are labelled (far left) as the logical $|0\rangle$, $|1\rangle$ and noncomputational $|N\rangle$ of the qubit.
- Other common notations are the parenthesis/bracket (left), ellipse (middle), and fusion tree (right) notations.

Chosen order: $\left|((\bullet,\bullet)_{\mathbf{1}},\bullet)_\tau\right\rangle, \left|((\bullet,\bullet)_\tau,\bullet)_\tau\right\rangle, \left|((\bullet,\bullet)_\tau,\bullet)_{\mathbf{1}}\right\rangle$
Opposite order: $\left|(\bullet,(\bullet,\bullet)_{\mathbf{1}})_\tau\right\rangle$, $\left|(\bullet,(\bullet,\bullet)_\tau)_\tau\right\rangle$, $\left|(\bullet,(\bullet,\bullet)_\tau)_{\mathbf{1}}\right\rangle$

$$\left|(\bullet,(\bullet,\bullet)_i)_k\right\rangle = \sum_{j=\mathbf{1},\tau} (F_k^{\tau\tau\tau})_{ij} \left|((\bullet,\bullet)_j,\bullet)_k\right\rangle$$

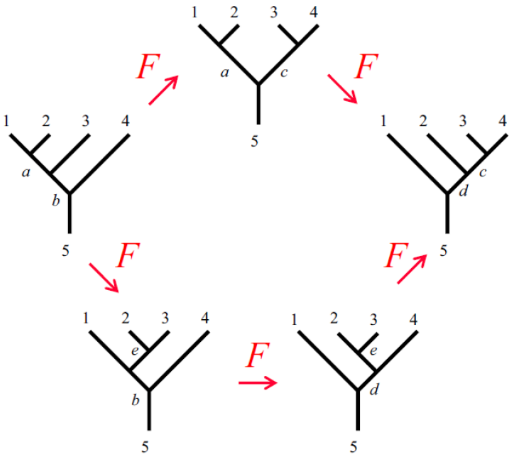
- $|N\rangle$: $\left|(\bullet,(\bullet,\bullet)_\tau)_{\mathbf{1}}\right\rangle = \left|((\bullet,\bullet)_\tau,\bullet)_{\mathbf{1}}\right\rangle$

$$F_{\mathbf{1}}^{\tau\tau\tau} = 1$$

- $|0\rangle$: $\left|(\bullet,(\bullet,\bullet)_{\mathbf{1}})_\tau\right\rangle = \sum_{j=\mathbf{1},\tau} (F_\tau^{\tau\tau\tau})_{\mathbf{1}j} \left|((\bullet,\bullet)_j,\bullet)_\tau\right\rangle$
- $|1\rangle$: $\left|(\bullet,(\bullet,\bullet)_\tau)_\tau\right\rangle = \sum_{j=\mathbf{1},\tau} (F_\tau^{\tau\tau\tau})_{\tau j} \left|((\bullet,\bullet)_j,\bullet)_\tau\right\rangle$

$$(F_\tau^{\tau\tau\tau}) = \begin{pmatrix} F_{\mathbf{11}} & F_{\mathbf{1}\tau} \\ F_{\tau\mathbf{1}} & F_{\tau\tau} \end{pmatrix} = \begin{pmatrix} 1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & -1/\phi \end{pmatrix}$$

4 τ particles: PENTAGON EQUATION (compatibility equation I)

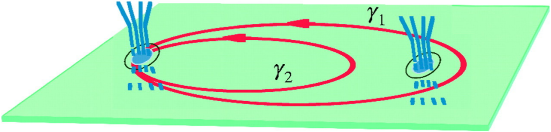


$$\left(F_5^{12c}\right)_d^a \left(F_5^{a34}\right)_c^b = \sum_e \left(F_d^{234}\right)_c^e \left(F_5^{1e4}\right)_d^b \left(F_b^{123}\right)_e^a$$

Braiding Fibonacci Anyons

Concept:

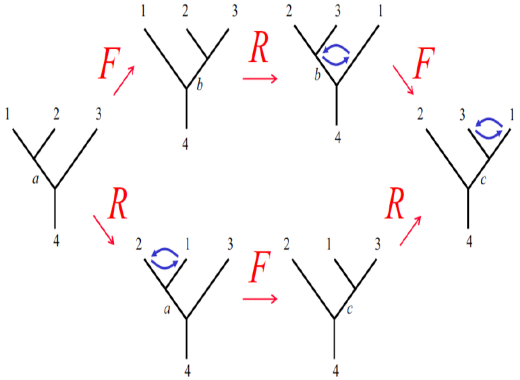
- Adiabatically braiding (winding) anyons around each other results in a unitary operation on a degenerate many-anyon Hilbert space
- Topological phase is imparted onto the anyons during the braid. As the anyons wind around each other, they pick up some phase due to the Aharonov-Bohm effect



2 τ particles: (rotation) R-matrix

$$R\,|(\bullet,\bullet)_{\mathbf{1}}\rangle = e^{-4\pi i/5}\,|(\bullet,\bullet)_{\mathbf{1}}\rangle$$
$$R\,|(\bullet,\bullet)_{\tau}\rangle = -e^{-2\pi i/5}\,|(\bullet,\bullet)_{\tau}\rangle$$

3 τ particles: Hexagon equation (compatibility equation II)
We can rotate before or after changing bases and we get the same result



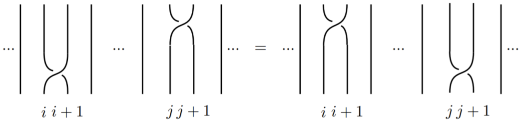
$$\sum_b \left(F_4^{231}\right)_c^b R_4^{1b} \left(F_4^{123}\right)_b^a = R_c^{13} \left(F_4^{213}\right)_c^a R_a^{12}$$

Braid Group & Its Representation

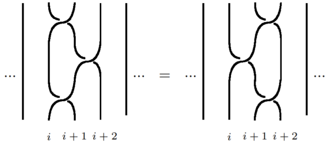
- **Braid group** on n particles (n strands):

$$B_n = \left\{ \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i \sigma_j = \sigma_j \sigma_i, \ |i-j| \geq 2, \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ 1 \leq i \leq n-1 \end{array} \right\},$$

where the braid group generators σ_i are the half right twists of the i-th strand about the $(i+1)$-th strand



$$\sigma_i \sigma_j = \sigma_j \sigma_i$$



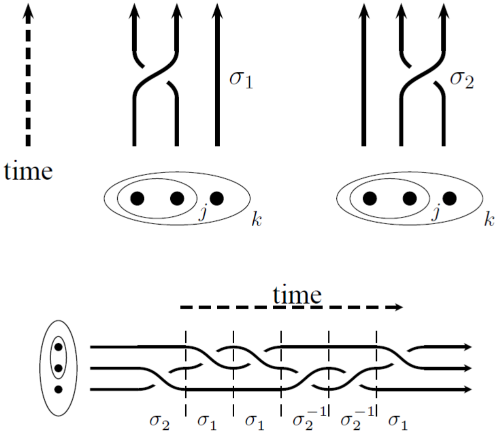
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

- Braiding the σ's produces a **representation** ρ_n,

$$\rho_n : B_n \to U(V_n)$$

from the braid group B_n on n strands into the unitary transformations of V_n (ground state subspace of $\mathcal{H}_n$)

Braid group (generators σ_1, σ_2):



Top: The two elementary braid operations σ_1 and σ_2 on three particles.
Bottom: Using these two braid operations and their inverses, an arbitrary braid on three strands can be built.
The braid shown here is written as $\sigma_2\sigma_1\sigma_1\sigma_2^{-1}\sigma_2^{-1}\sigma_1$.

Braid group representations $\rho(\sigma) = F^{-1}RF$:

$$\sigma_1 : \qquad \begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix} \to \underbrace{\left(\begin{array}{cc|c} e^{-4\pi i/5} & 0 & 0 \\ 0 & -e^{-2\pi i/5} & 0 \\ \hline 0 & 0 & -e^{-2\pi i/5} \end{array}\right)}_{\rho(\sigma_1)} \begin{pmatrix} |0\rangle \\ |1\rangle \\ |N\rangle \end{pmatrix}$$

$$\begin{aligned} \sigma_2 : \qquad |0\rangle &= F_{\mathbf{11}} \,|(\bullet, (\bullet, \bullet)_{\mathbf{1}})_\tau\rangle + F_{\tau\mathbf{1}} \,|(\bullet, (\bullet, \bullet)_\tau)_\tau\rangle \\ R\,|0\rangle &= e^{-4\pi i/5} F_{\mathbf{11}} \,|(\bullet, (\bullet, \bullet)_{\mathbf{1}})_\tau\rangle - e^{-2\pi i/5} F_{\tau\mathbf{1}} \,|(\bullet, (\bullet, \bullet)_\tau)_\tau\rangle \\ \rho(\sigma_2)\,|0\rangle &= ([F^{-1}]_{\mathbf{11}} e^{-4\pi i/5} F_{\mathbf{11}} - [F^{-1}]_{\mathbf{1}\tau} e^{-2\pi i/5} F_{\tau\mathbf{1}})\,|0\rangle \\ &\quad + ([F^{-1}]_{\tau\mathbf{1}} e^{-4\pi i/5} F_{\mathbf{11}} - [F^{-1}]_{\tau\tau} e^{-2\pi i/5} F_{\tau\mathbf{1}}\,|1\rangle \\ &= -e^{-\pi i/5}/\phi\,|0\rangle - ie^{-i\pi/10}/\sqrt{\phi}\,|1\rangle \end{aligned}$$

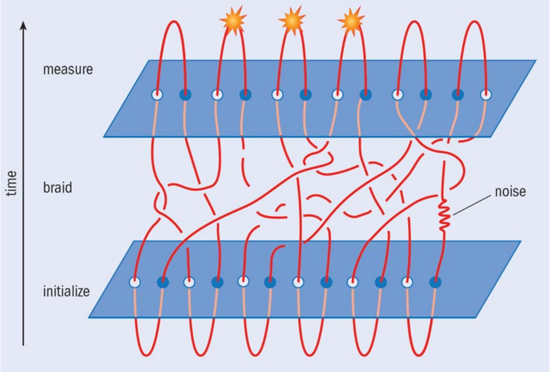
Similar results can be derived for the other two basis states to give the matrix

$$\rho(\sigma_2) = \left(\begin{array}{cc|c} -e^{-\pi i/5}/\phi & -ie^{-i\pi/10}/\sqrt{\phi} & 0 \\ -ie^{-i\pi/10}/\sqrt{\phi} & -1/\phi & 0 \\ \hline 0 & 0 & -e^{-2\pi i/5} \end{array}\right)$$

Universal Topological Quantum Computation

Basic idea to simulate quantum computation with anyons:

1. Choose a basis and restrict the Hilbert space
2. Braid the anyons together
3. Fuse the anyons at the end, and detect how they fuse in order to read the output of the system.



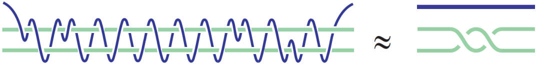
Single Qubit

2 τ particles – 2 states: $(|(\bullet,\bullet)_{\mathbf{1}}\rangle, |(\bullet,\bullet)_\tau\rangle)$ ✗

- Can never change $|(\bullet,\bullet)_{\mathbf{1}}\rangle$ to $|(\bullet,\bullet)_\tau\rangle$ by a single qubit operation (braid once)
- Amplitude that ends up in this state is known as "leakage error"

3 τ particles – 2 states: $(|((\bullet,\bullet)_{\mathbf{1}},\bullet)_\tau\rangle \equiv |0\rangle, |((\bullet,\bullet)_\tau,\bullet)_\tau\rangle \equiv |1\rangle)$
& 1 non-computational state: $(|((\bullet,\bullet)_\tau,\bullet)_{\mathbf{1}}\rangle \equiv |N\rangle)$ ✓

- Do single qubit operations with no leakage
- 3d Hilbert space for three particles
- $\rho(\sigma_1)$ and $\rho(\sigma_2)$ are block diagonal, never mix $|N\rangle$ with computational space $|0\rangle$ and $|1\rangle$



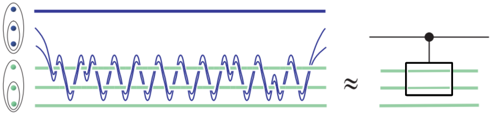
$$\sigma_2^3\sigma_1^2\sigma_2^{-4}\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^2\sigma_1^{-2}\sigma_2^4\sigma_1^{-2}\sigma_2^2\sigma_1^4\sigma_2^2\sigma_1^{-2}\sigma_2 \approx \sigma_1^2$$

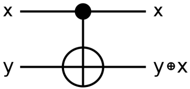
Constructing a braid on three strands moving only the blue particle has the same effect as interchanging the two green strands

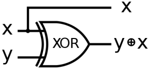
Multiple Qubits

- Able to perform single qubit operations & 2-qubit CNOT entangling gates
- Braiding together (physically "entangling"!) the particles/strands from two different qubits

$$|0\rangle = |((\bullet,\bullet)_1,\bullet)_\tau\rangle$$



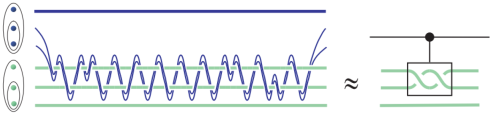




$$|1\rangle = |((\bullet,\bullet)_\tau,\bullet)_\tau\rangle$$

input		output	
x	y	x	y+x
$\vert 0\rangle$	$\vert 0\rangle$	$\vert 0\rangle$	$\vert 0\rangle$
$\vert 0\rangle$	$\vert 1\rangle$	$\vert 0\rangle$	$\vert 1\rangle$
$\vert 1\rangle$	$\vert 0\rangle$	$\vert 1\rangle$	$\vert 1\rangle$
$\vert 1\rangle$	$\vert 1\rangle$	$\vert 1\rangle$	$\vert 0\rangle$

input		output	
x	y	x	y+x
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0



Dense Images

- M. Freedman *et al.* (2000) generally guarantees that braids corresponding to any desired unitary operation exist on a 2-qubit Hilbert space.
- Braid group representations have **dense images** in the unitary group $\longrightarrow$ *a quantum state of Fibonacci anyons is said to be able to support universal quantum computation*
- More precisely, an arbitrary unitary transformation can be approximated, **up to a phase**, by a transformation in $\rho_n(B_n)$ to within any desired accuracy.
- **Projective group** $PU(V_n)$: a set of unitary transformations on V_n with two transformations identified if they differ only by a phase.
- $\rho_n(B_n)$ is **dense** in $PU(V_n)$, i.e. *the intersection of all closed sets containing* $\rho_n(B_n)$ *should simply be* $PU(V_n)$.

$SO(3)_2$ Chern-Simons (CS) Theory

Fibonacci anyons can be constructed as the $j = 0$ and $j = 1$ (quasi)particles in ("even" part of) $SU(2)_3$ CS theory satisfying the fusion rules of Fibonacci anyons

- For a modestly large number (≥ 7) of σ's, M. Freedman *et al.* (2000, 2001) shows that the braid group representations associated with $SU(2)_3$ CS theory are dense in $SU(V_n)$, and hence in $PU(V_n)$.
- Known as the **Jones representation**, which satisfies a key **two eigenvalue property (TEVP)**:
  image matrix of each braid generator σ_i under the Jones representation has exactly two distinct eigenvalues whose ratio is not ± 1.

$SU(2)_3$ Topological Modular Functor (TMF)

$$
\begin{tikzcd}
\text{MTC} \arrow[r, leftrightarrow] \arrow[dr] & \text{topological 2D MF} \arrow[r, leftrightarrow] \arrow[d, dotted, bend left] & \text{complex analytic 2D MF} \\
& \text{3D TQFT} \arrow[u, bend left] & \text{rational CFT} \arrow[u]
\end{tikzcd}
$$

MTC: modular tensor categories, MF: modular functors,
TQFT: topological quantum field theory, CFT: conformal field theory

[Bakalov and Kirilov, Lectures on Tensor Categories and Modular Functors]

Jones representation of braid group B_6 of 6 Fibonacci anyons

$$\Updownarrow$$

Representation of $SU(2)_3$ CSMF of braids at the fifth root of unity $q = e^{2\pi i/5}$

$SU(2)_3$ CSMF: build the "Chern-Simons5" (CS5) model which efficiently and fault tolerantly simulates the computations of an exact quantum circuit model

Density Theorem

Let $\rho := \rho_{[3,3]} \oplus \rho_{[4,2]} : B_6 \to U(5) \times U(8)$ be the Jones representation of the braid group B_6 at the 5-th root of unity $q = e^{\frac{2\pi i}{5}}$. Then the closure of the image of $\rho(B_6)$ in $U(5) \times U(8)$ contains $SU(5) \times SU(8)$.

- Let H be the closure of the image, $\overline{\text{image}(\rho)}$, of $\rho_{[3,3]}$ in $U(5)$ (or of $\rho_{[4,2]}$ in $U(8)$) which we will try to identify. H is a compact subgroup of $U(m)$ ($m = 5$ or 8) of positive dimension.
- Let φ be the induced m-dimensional faithful, irreducible complex representation of H, and let H_0 be the identity component of H.
- Actually want to show is that the derived group $[H_0, H_0]$ (or universal cover of the derived group $[\widetilde{H_0, H_0}]$) is actually $SU(m)$.

[M. Freedman *et al.* (2000), arXiv:quant-ph/0001108]
[M. Freedman *et al.* (2001), arXiv:math/0103200]

Sketch of proof:

- Using the Jones skein relation $q^{-1/2}\rho(\sigma_i) - q^{1/2}\rho(\sigma_i^{-1}) = q^{1/4} - q^{-1/4}$, one can assert that the fundamental representation of $U(m)$ restricted to H, $\varphi|_H$, has the TEVP $(q^{3/4}/-q^{1/4} \neq \pm 1)$.
- Further restricting to the identity component H_0, $\varphi|_{H_0}$ is *isotypic* (i.e. a direct sum of several copies of a single irreducible representation of H) and then *irreducible*. This implies that H_0 is *reductive* ($H_0 = H_0^{Der} Z(H_0)$, with $Z(H_0)$ the centre of H_0), so its derived group $H_0^{Der} := [H_0, H_0]$ is semisimple and, it can be argued, $\varphi|_{H_0^{Der}}$ still satisfies the TEVP and is still *irreducible*.
- A final (harmless) variation on H is to pass to the universal cover $H_0^{uc} := [\widetilde{H_0, H_0}]$. The representation $\varphi|_{H_0^{uc}}$ still has the TEVP and is still *irreducible*.
- *The sequence* $H \to H_0 \to [H_0, H_0] \to [\widetilde{H_0, H_0}]$ *did nothing!*
(Nothing changes beyond the first arrow, which may have eliminated some components of H *on which the determinant is a nontrivial root of unity).*

Sketch of proof (cont'd):

- The closed image of $\rho_{[3,3]}$ is $H \subset U(5)$, so our irreducible representation $\varphi|_{H_0^{uc}}$, coming from $U(5)$'s fundamental, is exactly 5-dimensional.
- From tables in [McKay and Patera (1981)],
  1. rank $= 1$: $(SU(2), 4w_1)$,
  2. rank $= 2$: $(Sp(4), w_2)$,
  3. rank $= 4$: $(SU(5), w_i),\ i = 1, 4,$

  where w_i is the fundamental weight. By examining the possible eigenvalues, we can exclude the first two cases as follows.
  1. Suppose $x \in SU(2)$ has eigenvalues α and β in w_1. Then under $4w_1$, it will have $\alpha^i\beta^j,\ i + j = 4\ (i, j \in \mathbb{Z}_+)$ as eigenvalues, which are too many (unless $\frac{\alpha}{\beta} = \pm 1$).
  2. Since 5 is odd, every element in the image has at least one real eigenvalue, with the others coming in reciprocal pairs. Again, there is no solution (unless there are two eigenvalues whose ratio is ± 1).
  3. Only possible pair: $H_0^{uc} \cong SU(5)$. Since φ is a faithful representation of H_0^{Der}, the image of H_0^{Der} is the same as that of H_0^{uc} which is $SU(5)$.

  Same eigenvalue analysis can be done for the 8-dimensional case for $\rho_{[4,2]}$, where one will get the irreducible representation $(SU(8), w_i),\ i = 1, 7$.
  Conclusion: $SU(5) \subset H_{U(5)} \subset U(5),\ SU(8) \subset H_{U(8)} \subset U(8)$.

Example: (5-dimensional) representations $\rho_{[3,3]}$ for braid generators σ_i

$$\rho_{[3,3]}(\sigma_1) = \begin{pmatrix} -1 & & & & \\ & q & & & \\ & & -1 & & \\ & & & q & \\ & & & & q \end{pmatrix},$$

$$\rho_{[3,3]}(\sigma_2) = \begin{pmatrix} \frac{q^2}{q+1} & -\frac{q\sqrt{[3]}}{q+1} & & & \\ -\frac{q\sqrt{[3]}}{q+1} & -\frac{1}{q+1} & & & \\ & & \frac{q^2}{q+1} & -\frac{q\sqrt{[3]}}{q+1} & \\ & & -\frac{q\sqrt{[3]}}{q+1} & -\frac{1}{q+1} & \\ & & & & q \end{pmatrix},$$

where $[3] = q + q^{-1} + 1$, and $\rho_{[3,3]}(\sigma_i)$ for $i = 3, 4, 5$ are similar.

[Funar (1998), arXiv:math/9804047]

Thanks!


HOW'S YOUR QUANTUM COMPUTER PROTOTYPE COMING ALONG?
GREAT!
THE PROJECT EXISTS IN A SIMULTANEOUS STATE OF BEING BOTH TOTALLY SUCCESSFUL AND NOT EVEN STARTED.
CAN I OBSERVE IT?
THAT'S A TRICKY QUESTION.